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S. E. Stepanov , ***I. I. Tsyganok***
Department of Mathematics,Financial University,Russia
s.e.stepanov@mail.ru, i.i.tsyganok@mail.ru
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On the differentiable sphere theorem for manifolds with Ricci curvatures bounded from above

In the present paper, we prove that if (M, g) is an n-dimensional $(n \geq 3)$ compact Riemannian manifold and if $Ric_{\max}(x) < n\, K_{\min}(x)$, where $K_{\min}(x) = \inf_{\pi \subset T_x M} K(\pi)$, $Ric_{\max}(x) = Ric_{X \in T_x M} Ric(X), K(\cdot)$ and $Ric(\cdot)$ are the sectional and Ricci curvatures of (M, g) respectively, then (M, g) is diffeomorphic to a spherical space form $\mathbb{S}^n/\Gamma$ where Γ is a finite group of isometries acting freely. In particular, if (M, g) is simply connected, then it is diffeomorphic to the Euclidian sphere $\mathbb{S}^n$.

Keywords: Riemannian manifold, sectional curvature, Ricci curvature, sphere theorem, spherical space form

1. Introduction: Sphere theorems

Let (M, g) be an n-dimensional $(n \geq 2)$ Riemannian manifold and $x \in M$. The *sectional curvature* in x of a 2-plane $\pi(x)$ spanned by an orthonormal basis $X, Y \in T_x M$ is given by $K(X, Y) = Rm(X, Y, X, Y)$ where Rm denotes the Riemannian curvature tensor.

Denote by $K_{\min}(x)$ the minimum of the sectional curvature of a Riemannian manifold (M, g) at a point $x \in M$. Since the unit sphere in $T_x M$ is a compact set, there exists a 2-plane $\pi(x) \subset T_x M$

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such that $K_{\min}(x) = K(\pi(x))$ — the sectional curvature in the direction of $\pi(x) \subset T_xM$. In other words,

$$K_{\min}(x) := \inf_{\pi(x) \subset T_xM} K(\pi(x)).$$

Since (M, g) is a compact manifold, we can define a scalar invariant $K_{\min} := \inf_{x \in M} K(x)$ of (M, g).

In a similar way we can define the maximum of the sectional curvature of (M, g) at a point $x \in M$. Namely, we let $K_{\max}(x) := \sup_{\pi(x) \subset T_xM} K(\pi(x))$.Next, to determine $K_{\max}(x)$ we use the condition $K_{\max} := \sup_{x \in M} K(x)$.

Berger proved in [1] the following "topological sphere theorem": a compact, simply connected Riemannian manifold (M, g) whose sectional curvatures satisfy the condition $0 < K_{\min} \leq K(x) \leq K_{\max} = 4K_{\min}$ at an arbitrary point $x \in M$, is either homeomorphic to $\mathbb{S}^n$ or isometric to a compact symmetric space of rank one.

On other hands, Brendle and Shoen proved in [2] "the differential sphere theorem": if a compact, simply connected Riemannian manifold (M, g) is not locally symmetric space and its sectional curvatures satisfy the condition

$$0 \leq K_{\min}(x) \leq K(x) \leq K_{\max}(x) = 4K_{\min}(x)$$

at an arbitrary point $x \in M$, then (M, g) is diffeomorphic to a spherical space form.

Contractions of sectional curvature leads to the *Ricci curvature* Ric. Namely, it can be show that

$$Ric(X) = \sum_{a=2}^{n} K(X, e_a)$$

for given any unit vector $X \in T_xM$, pick an orthonormal basis $\{e_1, \dots, e_n\}$ for T_xM such that $X = e_1$. Therefore, the *Ricci tensor* Ric can be interpreted as the sum of sectional curvatures of planes spanned by a unit vector X in the tangent space and other elements of an orthonormal basis to which X belongs. In this case, we can obtain the well-known double inequality

$$(n-1)K_{\min}(x) \ \leq Ric(X) \leq (n-1)K_{\max}(x) \qquad (1)$$

where $X \in T_xM$ is an arbitrary unit vector at $x \in M$. Since the unit sphere in T_xM at an arbitrary point $x \in M$ is a compact set, there exists $Ric_{\min}(x) \coloneqq \inf_{X \in T_xM} Ric(X)$.

Xu and Gu proved in [3] the following "differentiable sphere theorem": a compact Riemannian manifold whose Ricci curvature and sectional curvatures satisfy the inequality

$$Ric_{\min}(x) > \big((n-1) - 6/5\big)K_{\max}(x) \qquad (2)$$

for any unit vector $X \in T_xM$ at an arbitrary point $x \in M$ is diffeomorphic to a spherical space form $\mathbb{S}^n/\Gamma$, where Γ is a finite group of isometries acting freely. In particular, if (M, g) is simply connected, then (M, g) is diffeomorphic to the standard Euclidian n-sphere $\mathbb{S}^n$.

From (1) and (2) we obtain the double inequality

$$(n-1)K_{\max}(x) - 6/5\, K_{\max}(x) < Ric(X) \leq (n-1)K_{\max}(x),$$

where $X \in T_xM$ is an arbitrary unit vector at $x \in M$. At the same time, one can obtain from (1) and (2) that the Ricci curvature $Ric(\cdot) > 0$ at each point $x \in M$. Therefore, the above theorem is called "the differentiable sphere theorem for manifolds with positive Ricci curvature" (see [3]).

2. New version of the Sphere theorem

Since the unit sphere in T_xM at an arbitrary point $x \in M$ is a compact set, there exists $Ric_{\max}(x) \coloneqq \inf_{X \in T_xM} Ric(X)$. Then we, in turn, will be able to prove our "differentiable sphere theorem" for Riemannian manifolds with Ricci curvatures bounded from above.

Theorem. *Let (M, g) be an n-dimensional $(n \geq 3)$ compact Riemannian manifold and Ric be its Ricci tensor satisfying the inequality*

$$Ric_{\max}(x) < nK_{min}(x) \qquad (3)$$

at each point $x \in M$. Then (M, g) is diffeomorphic to a spherical space form $\mathbb{S}^n/\Gamma$. In particular, if (M, g) is simply connected, then (M, g) is diffeomorphic to the Euclidian sphere $\mathbb{S}^n$.

Proof. First, from (1) and (3) we obtain the double inequality

$$(n-1)K_{\min}(x) \le Ric(X) < (n-1)K_{min}(x) + K_{min}(x)$$

where $X \in T_x M$ is an arbitrary unit vector at $x \in M$. At the same time, one can obtain from (1) and (3) that the sectional curvature $K(\cdot) > 0$ at each point $x \in M$.

Second, we recall the definition of *the curvature operator of the second kind* (see [4]). Namely, the Riemann curvature tensor Rm induces an algebraic curvature operator $\overset{\circ}{R}: S_0^2 M \to S_0^2 M$ for the space $S_0^2 M$ of trace-free symmetric two-tensor fields (see, for example, [4]). The symmetries of Rm imply that $\overset{\circ}{R}$ is a selfadjoint operator, with respect to the point-wise inner product on $S_0^2 M$. In this case, $\overset{\circ}{R}$ is called as *the curvature operator of the second kind* (see [4]). Moreover, the map $\overset{\circ}{R}: S_0^2 M \to S_0^2 M$ induces a bilinear form $\Phi: S_0^2 M \times S_0^2 M \to \mathbb{R}$, which is defined by the equlity $\Phi(\varphi) = g\left(\overset{\circ}{R}(\varphi), \varphi\right)$ for an arbitrary $\varphi \in S_0^2 M$. Accordingly, we will say that $\overset{\circ}{R} > 0$ if the eigenvalues of $\overset{\circ}{R}$ as a bilinear form on $S_0^2 M$ are positive.

Thirdly, we will prove our theorem. The bilinear form Φ satisfies the inequality (see [5])

$$\Phi(\varphi) \ge nK_{\min}(x)\|\varphi\|^2 - R_{ij}\varphi^{ik}\varphi_k^j. \tag{4}$$

for the local components φ^{ik} and φ_k^j of an arbitrary $\varphi \in S_0^2(T_x M)$ at each point $x \in M$. In addition, the following inequality $R_{ij}\varphi^{ik}\varphi_k^j \le Ric_{\max}(x)\|\varphi\|^2$ holds. Then from (4) we deduce the inequality

$$\Phi(\varphi) \ge \left(nK_{\min}(x) - Ric_{\max}(x)\right)\|\varphi\|^2. \tag{5}$$

In this case, we conclude from (5) that $\overset{\circ}{R} > 0$ if $Ric_{\max}(x) < n\,K_{\min}(x)$ at each point $x \in M$. At the same time, we know from [3] that if (M, g) be an n-dimensional ($n \ge 3$) compact Riemannian manifold such that $\overset{\circ}{R}$ is strictly positive, then M is diffeomorphic to a spherical space form $\mathbb{S}^n/\Gamma$. In this case, if (M,g) is simply connected, then (M,g) is diffeomorphic to the Euclidian n-sphere $\mathbb{S}^n$. Therefore, our theorem holds.

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С. Е. Степанов, И. И. Цыганок

Финансовый университет при Правительстве РФ, Россия

s.e.stepanov@mail.ru, i.i.tsyganok@mail.ru

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Теоремы о дифференцируемых сферах для многообразий с ограниченными сверху кривизнами Риччи

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В представленной статье мы доказываем, что если (M,g) — это -мерное $(n \geq 3)$ компактное риманово многообразие и если $Ric_{\max}(x) < nK_{\min}(x)$, где $K_{\min}(x) = \inf_{\pi \subset T_xM} K(\pi)$, $Ric_{\max}(x) =$

$= Ric_{X \in T_x M} Ric(X)$, $K(\cdot)$ и $Ric(\cdot)$ — секционная кривизна и кривизна Риччи многообразия (M, g), то оно будет диффеоморфным сферической пространственной форме $\mathbb{S}^n/\Gamma$. В частности, если (M, g) односвязное, то оно диффеоморфно евклидовой сфере $\mathbb{S}^n$.

Ключевые слова: риманово многообразие, секционная кривизна, кривизна Риччи, теорема о сфере, сферическая пространственная форма

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