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On conformal transformations of metrics of Riemannian paracomplex manifolds

A $2n$-dimensional differentiable manifold M with $O(n,\mathbf{R})\times O(n,\mathbf{R})$-structure is a Riemannian almost paracomplex manifold. In the present paper, we consider conformal transformations of metrics of Riemannian paracomplex manifolds. In particular, a number of vanishing theorems for such transformations are proved using the Bochner technique.

Keywords: almost paracomplex manifold, conformal transformations, Bochner technique, mixed scalar curvature.

§ 1. Introduction and results

A Riemannian almost product structure on an m-dimensional Riemannian manifold (M,g) is a (1, 1)-tensor field J on M such that $J^2=\mathrm{id}$ and $g(J,J)=g$. The triplet (M,J,g) is called a Riemannian almost product manifold (see [1]). As a result, at every point $x\in M$, the horizontal subspace H_x and the vertical subspace V_x of the tangent space T_xM that correspond to the eigenvalues -1 and $+1$ of the tensor J must be orthogonal. A Riemanni-

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an almost paracomplex manifold is a Riemannian almost product manifold (M, J, g) such that $trace J = 0$ (see [2]). In this case, the two eigenbundles H and V of the tangent bundle TM have the same rank, i. e., $\dim H_x = \dim V_x$ at any point $x \in M$. Note that the dimension of an almost paracomplex manifold (M, J, g) is necessarily even, i. e., $m = 2n$.

A Riemannian almost paracomplex structure on $2n$-dimensional differentiable manifold M may alternatively be defined as a $O(2n, \mathbf{R})$-structure on M with structural orthogonal group $O(2n, \mathbf{R}) = O(n, \mathbf{R}) \times O(n, \mathbf{R})$. This structure is the antipode of an *almost complex structure* (see [3]).

There are three kinds of sectional curvature for a Riemannian almost product manifold (M, J, g): horizontal, vertical, and mixed. The mixed plane is spanned by two vectors such that the first vector is horizontal and the second vector is vertical at an arbitrary $x \in M$. Mixed curvatures stand for the sectional curvatures of mixed planes. This concept has a long history and many applications (see, e. g., [4] and [5]).

Let $\{e_1, \dots, e_p\}$ be an adapted local orthonormal frame of H and $\{e_{p+1}, \dots, e_m\}$ be an adapted local orthonormal frame of V. Then *mixed scalar curvature* of a Riemannian almost product manifold (M, J, g) is an averaged mixed sectional curvature, i. e., the following function on M:

$$s_{mix} = \sum_{i=1,\dots,p} \; \sum_{\alpha=p+1,\dots,m} g\left(R(e_i, e_\alpha)e_a, e_i\right),$$

where R is the curvature tensor of the metric g.

We will assume below that there is a *conformal transformation* $f : (M, J, g) \to (M, J, \bar{g})$ of a Riemannian almost paracomplex manifold (M, J, g) such that the metrics g and $\bar{g}$ are conformally equivalent, that is, the following ordinary condition $f^{*}\overline{g} = e^{2\sigma} g$ for some smooth scalar function σ is satisfied (see [6, p. 269]). This transformation preserves the angles between any pair of curves (see [6, p. 267]). Therefore, the almost paracomplex structure of (M, J, g) is also preserved. In particular, if σ is constant then *f* is called *homothetic transformation*. In this case, the following theorem holds.

Theorem 1. *Let* (M, J, g) *be a 2n-dimensional parabolic Riemannian almost paracomplex manifold with the mixed scalar curvature* $s_{\text{mix}} \le 0$. *Then there are no non-homothetic conformal transformations of the metric g such that* $\overline{s}_{\text{mix}} \ge 0$.

A complete Riemannian manifold of finite volume is an example of *parabolic manifolds* (see [7]). Using this fact, we can formulate a corollary of Theorem 1.

Corollary 1. *Let* (M, J, g) *be a 2n-dimensional complete noncompact Riemannian almost paracomplex manifold of finite volume. If the mixed scalar curvature* $s_{\text{mix}} \le 0$, *then there are no nonhomothetic conformal transformations of the metric g such that* $\overline{s}_{\text{mix}} \ge 0$.

In particular, if *M* is a compact manifold, then the following statements hold.

Theorem 2. *Let* (M, J, g) *be a 2n-dimensional compact Riemannian almost paracomplex manifold with the mixed scalar curvature* $s_{\text{mix}} \le 0$ *(resp.* $s_{\text{mix}} \ge 0$*). Then there are no non-homothetic conformal transformations of the metric g such that* $\overline{s}_{\text{mix}} \ge 0$ *(resp.* $\overline{s}_{\text{mix}} \le 0$*).*

Theorem 3. *Let* (M, J, g) *and* $(M, J, \overline{g})$ *be two 2n-dimensional compact almost paracomplex manifolds with conformally equivalent metrics g and* $\overline{g}$. *If their mixed scalar curvatures* s_{mix} *and* $\overline{s}_{\text{mix}}$ *are not equal to zero everywhere on M, then these curvatures have the same sing.*

§ 2. Proofs of Theorems

Let (M, J, g) be a 2*n*-dimensional Riemannian almost para-complex manifold. We will assume that *f* is a conformal transformation such that $\bar{g} = u^{4/(m-2)} g$, i. e., $\sigma = (2/(m-2)) \ln u$, for some smooth function $u > 0$ and *f*-adjusted common coordinates on *M* (see [5, p. 269]). In this case, we get (see [8])

$$\Delta u = \frac{m-2}{m} \left(u\, s_{mix} - u^{\frac{m+2}{m-2}} \bar{s}_{mix} \right), \tag{1}$$

where $\Delta u = div(grad\, u)$ is the *Beltrami Laplacian*. Therefore, if $s_{\mathrm{mix}} \le 0$ and $\bar{s}_{\mathrm{mix}} \ge 0$ then $\Delta u \le 0$, i. e., *u* is *a superharmonic function*. We recall here that a complete Riemannian manifold is *parabolic* if it does not admit non-constant positive superharmonic functions (see e. g., [9, p. 313]). Using this fact, we prove our Theorem 1.

If a manifold *M* is compact, then integrating (1) over (*M, g*) and using *Green theorem* $\int_M \Delta u\, dv_g = 0$ gives

$$\int_M u\, s_{mix}\, dv_g = \int_M u^{\frac{m+2}{m-2}} \bar{s}_{mix}\, dv_g\ . \tag{2}$$

Using (2), we prove our Theorems 2 and 3.

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О конформных преобразованиях метрик римановых паракомплексных многообразий

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$2n$-мерное дифференцируемое многообразие M с $O(n,\mathbf{R})\times O(n,\mathbf{R})$-структурой называется римановым паракомплексным многообразием. В статье изучаются конформные преобразования метрик паракомплексных многообразий. В частности, доказывается с помощью техники Бохнера ряд теорем исчезновения для таких преобразований.

Ключевые слова: почти паракомплексное многообразием, конформные преобразования, техника Бохнера, смешанная скалярная кривизна.

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