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Pointwise orthogonal splitting of the space of *TT*-tensors

In the present paper we consider pointwise orthogonal splitting of the space of well-known *TT*-tensors on Riemannian manifolds. Tensors of the first subspace belong to the kernel of the Bourguignon Laplacian, and the tensors of the second subspace belong to the kernel of the Sampson Laplacian. We give examples and prove Liouville-type non-existence theorems of these tensors.

Keywords: Riemannian manifold, *TT*-tensor, Liouville-type non-existence theorems, sectional curvature

Introduction

Let (M, g) be an n-dimensional $(n \geq 2)$ Riemannian manifold with the Levi-Civita connection ∇. By S^2M we understand the vector bundle of symmetric bilinear differential two-forms. We define the *divergence of symmetric two-tensors fields* $\delta: C^\infty S^2M \to$ $\to C^\infty T^*M$ by the formula $\delta := -\,trace_g \circ \nabla$ (see [1, p. 35]).

We recall that a symmetric divergence free and traceless covariant two-tensor (transverse-trace free tensor) field is called TT-tensor (see, for instance, [2]). The vector space of TT-tensors φ^{TT} is defined by the condition

$$S^{TT}(M) := \{\varphi \in S^2M \mid \delta\,\varphi = 0, trace_g\varphi = 0\}.$$

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Any TT-tensor is denoted by φ^{TT} (see [2]). As a consequence of a result of Bourguignon — Ebin — Marsden (see [1, p. 132; 2]) the space of TT-tensors is an infinite-dimensional vector space on any closed Riemannian manifold (M, g). Such tensors are of fundamental importance in stability analysis in General Relativity (see, for instance, [3]) and in Riemannian geometry (see, for instance, [1, p. 346—347; 2]). A simple example of TT-tensors is the Ricci tensor of a Riemannian manifold of zero scalar curvature.

The tangent space T_xM at any $x \in M$ is an n-dimensional Euclidian vector space E with the orthogonal group $O(n, \mathbb{R})$. We consider the space

$$\mathfrak{S}(E) = \{G \in E^* \otimes S_0^2 E \mid G_{12}(c) = 0\},$$

where $S_0^2 E$ is the space of trace-free symmetric two-tensors on E and $G_{12}(c) = \sum_{k=1}^{n} G(e_k, e_k, c)$ for an orthonormal basis $\{e_1, \dots, e_n\}$ and an arbitrary c in E. By [4], the tensor space $\mathfrak{S}(E)$ has the orthogonal splitting $\mathfrak{S}(E) = \mathfrak{S}_1(E) \oplus \mathfrak{S}_2(E)$ for two subspaces irreducible with respect to the action the orthogonal group $O(n, \mathbb{R})$:

$$\mathfrak{S}_1(E) = \{\Phi \in \mathfrak{S}(E) \mid \Phi(a, b, c) = \Phi(b, a, c)\},$$

$$\mathfrak{S}_2(E) = \{\Phi \in \mathfrak{S}(E) \mid \Phi(a, b, c) + \Phi(b, c, a) + \Phi(c, a, b) = 0\}$$

for arbitrary *a, b, c* in E. Then the tensor field $\nabla\varphi^{TT}$ on (M, g) is a section of the vector bundle $\mathfrak{S}(TM)$, the fiber of which at each point $x \in M$ is the space $\mathfrak{S}(E)$. As a consequence, we obtain a pointwise decomposition of $\nabla\varphi^{TT}$ into a sum of the tensor fields corresponding to the pointwise irreducible components of the action of the group $O(n, \mathbb{R})$. This decomposition of $\nabla\varphi^{TT}$ determines a rough classification of TT-tensors, where the first class $\mathfrak{S}_1$ consists of TT-tensors for which their covariant derivatives are sections of $\mathfrak{S}_1(TM)$ and the second class $\mathfrak{S}_2$ consists of TT-tensors for which their covariant derivatives are sections of $\mathfrak{S}_2(TM)$.

1. The first class of transverse-trace free tensors

Suppose $\varphi^{TT} \in \mathfrak{S}_1$, then it satisfies the differential equation

$$(\nabla_X \varphi^{TT})(Y, Z) = (\nabla_Y \varphi^{TT})(X, Z)$$

for any $X, Y, Z \in TM$. In this case, the condition $trace_g \varphi^{TT} = 0$ takes the form $\delta\, \varphi^{TT} = 0$. It follows that the tensor field φ^{TT} is a *Codazzi tensor* with zero trace and zero divergence.

A simple example of $\varphi^{TT} \in \mathfrak{S}_1$ is the second fundamental form of the minimal hypersurface of a Riemannian manifold of constant curvature (see [1, p. 436]). On the other hand, the geometry of manifolds bearing Codazzi tensor fields is described in detail in the literature (see, for example, the survey in [1, p. 590—598]). In turn, we can formulate the following local result.

Theorem 1. *Let (M, g) be a Riemannian manifold of constant curvature C. Then a TT-tensor $\varphi^{TT} \in \mathfrak{S}_1$ has the form*

$$\varphi^{TT} = Hess\,(f) + C \cdot f\, g$$

where $f \in C^2 M$ is a solution of the equation $\Delta\, f + n\, C\, f = 0$ for the Beltrami Laplacian on functions $\Delta := trace_g \nabla^2$.

Proof. A Codazzi tensor φ on (M, g) with constant curvature C has the form (see [1, p. 436])

$$\varphi = Hess\,(f) + C \cdot f\, g$$

for the C^2-function f on (M, g). If $\varphi = \varphi^{TT}$, then it is a solution of the equations $\Delta\, f + n\, C\, f = 0$ and $\bar{\Delta}\, df = C\, df$, because $trace_g \varphi = 0$ and $\delta\, \varphi = 0$, respectively. Recall that $\bar{\Delta} := \delta \circ \nabla$ is *the rough Laplacian* (see [1, p. 52]). In addition, it is easy to prove that the second equation above is a consequence of the first equation.

J.P. Bourguignon (see [1, p. 355; 5, p. 273]) constructed the Laplacian $\Delta_B : C^\infty S^2 M \to C^\infty S^2 M$ by the formula

$$\Delta_B := d\,\delta + \delta\, d$$

where

$$d\,\varphi\,(X,Y,Z) := (\nabla_X \varphi)(Y,Z) = (\nabla_Y \varphi)(X,Z)$$

for arbitrary $X, Y, Z \in TM$ (see [1, p. 355]). In turn, $\varphi \in C^\infty S^2 M$ is called *harmonic* if it belongs to the kernel of Δ_B. Therefore, a Codazzi tensor with constant trace and, in particular, with zero trace is harmonic (see [6, p. 350]). In turn, if (M, g) is a closed manifold and a harmonic symmetric bilinear form φ is given in a global way on (M, g) then $\varphi \in \ker \Delta_B$ (see [7]). Furthermore, an arbitrary $\varphi \in \ker \Delta_B$ on a closed Riemannian manifold (M, g) with nonnegative sectional curvature $K(\sigma)$ is parallel and if $K(\sigma) > 0$ at some point, then φ is a constant multiple of g (see also [7]). Using the above, we can formulate our theorem.

Theorem 2. *An arbitrary TT-tensor $\varphi^{TT} \in \mathfrak{S}_1$ on a closed Riemannian manifold (M, g) with nonnegative sectional curvature $K(\sigma)$ is parallel. Moreover, if $K(\sigma) > 0$ at some point, then φ^{TT} is a zero-tensor.*

It is well known that a Riemannian symmetric space of compact type is a compact (without boundary) Riemannian manifold with nonnegative sectional curvature (see [6, p. 387]). One can state the following assertion.

Corollary 1. *An arbitrary TT-tensor $\varphi^{TT} \in \mathfrak{S}_1$ on a Riemannian symmetric space of compact type (M, g) is parallel. Moreover, if the holonomy group* $\mathrm{Hol}(g)$ *of the space (M, g) is irreducible, then $\varphi^{TT} \equiv 0$.*

On the other hand, we proved the following proposition (see [7; 8]): if the sectional curvatures of a connected complete noncompact Riemannian manifold (M, g) are everywhere nonnegative, then there exists no nonzero Codazzi tensor $\varphi \in C^\infty S_0^p M$, $p \geq 2$, such that $\int_M \|\varphi\|^q d\mathrm{vol}_g < +\infty$ for at least one $q \in (0, +\infty)$. Therefore, we have the following theorem.

Theorem 3. *Let (M, g) be a connected complete noncompact Riemannian manifold with nonnegative sectional curvature. Then there is no a nonzero TT-tensor $\varphi^{TT} \in \mathfrak{S}_1$ such that $\int_M \|\varphi^{TT}\|^q d\mathrm{vol}_g < +\infty$ for at least one $q \in (0, +\infty)$.*

2. The second class of transverse-trace free tensors

Now suppose that $\varphi^{TT} \in \mathfrak{S}_2$, then it satisfies the differential equation

$$(\nabla_X \varphi^{TT})(Y,Z) + (\nabla_Y \varphi^{TT})(Z,X) + (\nabla_Z \varphi^{TT})(X,Y) = 0 \quad (1)$$

for any $X, Y, Z \in TM$. In this case, the condition $trace_g \varphi^{TT} = 0$ takes the form $\delta\, \varphi^{TT} = 0$. From (1) it follows that the tensor field φ^{TT} is a *symmetric Killing two-tensor* with zero trace and zero divergence. The geometry of manifolds bearing symmetric Killing tensor fields is described in detail in the literature (see, for example, [9; 10]). For a Riemannian manifold of constant curvature, we can formulate the following local result.

Theorem 4. *Let (M, g) be a Riemannian manifold of constant curvature, then a TT-tensor $\varphi^{TT} \in \mathfrak{S}_2$ has the form*

$$\varphi_{ij}^{TT} = e^{2\omega}\left(A_{ijkl}\, x^k x^l + B_{ijk} x^k + C_{ij}\right) \quad (2)$$

for $\omega = (n+1)^{-1} \ln(\det g)$ with respect to a local coordinate system $\{x^1, \dots, x^n\}$ of (M, g). The coefficients A_{ijkl}, B_{ijk} and C_{ij} of (2) *are constant symmetric with respect to the first two subscripts and satisfying the identities*

$$A_{ijkl} + A_{jkil} + A_{kijl} = 0; \quad (3)$$

$$B_{ijk} + B_{jki} + B_{kij} = 0; \quad (4)$$

$$g^{ij} A_{ijkl} = g^{ij} B_{ijk} = g^{ij} \mathrm{C}_{ij} = 0 \quad (5)$$

for $i, j, k, l = 1, \dots, n$.

Proof. According to [11] and [12], if (M, g) is a Riemannian manifold of constant curvature, then the general solution of the equation $(\nabla_X \varphi)(X, X) = 0$ for any $X \in TM$ has the form (2) for $\omega = (n+1)^{-1} \ln(\det g)$ and the constants A_{ijkl}, B_{ijk}, C_{ij} which are symmetric with respect to the first two subscripts and satisfying (3) and (4). In this case, the condition $trace_g \varphi = 0$ takes the form (5). Therefore, the equalities (2)—(5) describe the solution of (1).

We proved that every divergence-free Killing tensor $\varphi \in C^{\infty}S^2M$ on a closed Riemannian manifold (M,g) with nonpositive sectional curvature $K(\sigma)$ is parallel and if $K(\sigma) < 0$ at some point, then φ is a constant multiple of g (see [13]). This implies the following result.

Theorem 5. *An arbitrary TT-tensor* $\varphi^{TT} \in \mathfrak{S}_2$ *on a closed Riemannian manifold* (M,g) *with nonpositive sectional curvature* $K(\sigma)$ *is parallel. Moreover, if* $K(\sigma) < 0$ *at some point, then* φ^{TT} *is a zero-tensor.*

If $\varphi \in C^{\infty}S^2M$ is a traceless symmetric Killing tensor on (M,g) and $\Delta_S := \delta\delta^* - \delta^*\delta$ is the *Sampson Laplacian*, then φ belongs to $\ker \Delta_S$ (see [1, p. 356; 24]). We recall here that the symmetric derivative $\delta^*: C^{\infty}S^2M \to C^{\infty}S^3M$ is defined by the equation

$$\delta^*\varphi\,(X,Y,Z) := (\nabla_X\varphi)(Y,Z) + (\nabla_Y\varphi)(Z,X) + (\nabla_Z\varphi)(Y,Z)$$

for arbitrary $X,Y,Z \in TM$ (see [1, p. 35, 356]). At the same time, in [14] was proved that there is no a nonzero symmetric two-tensor field $\varphi \in \ker \Delta_S$ on a simply connected complete Riemannian manifold (M,g) such that $\int_M \|\varphi\|^q d\mathrm{vol}_g < +\infty$ for at least one $q \in (0,+\infty)$. Note that such Riemannian manifold is diffeomorphic to $\mathbb{R}^n$ and has an infinite volume.Using the above, we can formulate the following theorem.

Theorem 6. *There is no a nonzero TT-tensor* $\varphi^{TT} \in \mathfrak{S}_2$ *on a simply connected complete Riemannian manifold* (M,g) *of nonpositive sectional curvature such that* $\int_M \|\varphi^{TT}\|^q d\mathrm{vol}_g < +\infty$ *for at least one* $q \in (0,+\infty)$.

An example of such a manifold is a Riemannian symmetric manifold (M,g) of non-compact type, since it is simply connected and has nonpositive curvature.

Corollary 2. *There is no a nonzero TT-tensor* $\varphi^{TT} \in \mathfrak{S}_2$ *on a Riemannian symmetric manifold of non-compact type* (M,g) *such that* $\int_M \|\varphi^{TT}\|^q d\mathrm{vol}_g < +\infty$ *for at least one* $q \in (0,+\infty)$.

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Поточечное ортогональное расщепление пространства *TT*-тензоров

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В статье рассматривается ортогональное расщепление пространства известных *TT*-тензоров на римановых многообразиях. Тензоры первого подпространства принадлежат ядру лапласиана Бургиньона, а тензоры второго подпространства принадлежат ядру лапласиана Сэмпсона. Приводятся примеры и доказываются теоремы Лиувилля о несуществовании этих тензоров.

Ключевые слова: риманово многообразие, *TT*-тензор, теоремы несуществования лиувиллевского типа, секционная кривизна

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