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Congruences of quadrics in three-dimensional projective space associated with pair of surfaces

Two-parametric family (congruence) K_2 of quadrics Q in three-dimensional projective space P_3 is investigated, possessing the following properties: on each quadric $Q \in K_2$ there are two different focal points A_1 and A_2 at which focal tangents intersect at one point A_0 and are the asymptotic tangents of the surface (A_0), and the tangents to the curves on the surface (A_i) that corresponds the focal curves on the surface (A_j) ($i, j, k = 1, 2; i \neq j$) also intersect at one point A3 and are the asymptotic tangents of the surface (A_3), mareover the asymtotic curves that envelop A_0A_i and A_3A_j are correspond, and A_0 and A_3 are polar conjugated.

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Decomposition theorems of conformal Killing forms on totally umbilical submanifolds

A Riemannian manifold of positive curvature operator has been studied from many directions. It is well known, that an n-dimensional closed Riemannian manifold with positive curvature operator $\mathfrak{R}$ is a spherical space form and its Betti numbers $b_1(M')$, …, $b_{n-1}(M')$ are zero. In addition, we proved that on an

n-dimensional closed Riemannian manifold (M, g) with positive curvature operator $\Re$ an arbitrary conformal Killing r-form ω is uniquely decomposed in the form $\omega' + \omega''$ where ω' is a Killing r-form and ω'' is a closed conformal Killing r-form on (M, g) for all $r = 1, \ldots, n-1$. In the present paper we prove three decomposition theorems of conformal Killing forms on totally umbilical submanifolds in Riemannian manifolds.

Key words: conformal Killing form, decomposition theorem, totally umbilical submanifold, Riemannian manifolds.

1. Let R be the covariant curvature tensor of a Riemannian manifold (M, g). The relation (see: [1, p. 36])

$$g(\Re(X \wedge Y), V \wedge W) = R(X \wedge Y, V \wedge W) = R(X, Y, W, V)$$

defines a self-adjoint symmetric operator $\Re : \Lambda^2 M \to \Lambda^2 M$. This operator is called the *curvature operator* of (M, g).

Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis in T_xM at an arbitrary point $x \in T_xM$ such that $e_i \wedge e_j$ diagonalize the curvature operator $\Re(e_i \wedge e_j) = \lambda_{ij}(x)\, e_i \wedge e_j$. We say that a Riemannian manifold (M, g) has positive curvature operator if all its eigenvalues are positive. Note that this definition is invariant, because it does not depend on the choice of the basis $\{e_1, \ldots, e_n\}$ at $x \in T_xM$. If the operator curvature $\Re$ is positive-definite at each point $x \in M$ then we say that $\Re$ is *positive-definite* on the manifold (M, g).

Next, we say that the positive curvature operator $\Re$ is *bounded from below* on (M, g) if all the eigenvalues of $\Re$ greater than or equal to some positive number λ at all points of (M, g). Side by side, we recall that if the positive curvature operator $\Re$ of a simply connected closed manifold (M, g) is bounded from below then (M, g) is homeomorphic to a sphere (see: [2]).

The normalized quadratic form $sec(\pi) = \dfrac{g(\Re(X_x \wedge Y_x), X_x \wedge Y_x)}{g(X_x \wedge Y_x, X_x \wedge Y_x)}$ is called the *sectional curvature of two-plane*

$\pi = \mathrm{span}\{X_x, Y_x\} \subset T_x M$. In addition, we note (see: [1], p. 63) that if $\{e_1, \ldots, e_n\}$ is an orthogonal basis for T_xM such that $e_i \wedge e_j$ diagonalize the curvature operator $\Re(e_i \wedge e_j) = \lambda_{ij}(x)\, e_i \wedge e_j$ then for any two-plane π in T_xM we have $sec(\pi) \in \lfloor min\, \lambda_{ij}(x), max\, \lambda_{ij}(x) \rfloor$. Moreover (see: [3]), if the eigenvalues of $\Re$ are $\lambda_{ij}(\tilde{o}) \geq \lambda(\tilde{o}) > 0$ then the sectional curvatures are $\lambda_{ij}(\tilde{o}) \geq \lambda(\tilde{o})/2 > 0$ at $x \in M$. Therefore, if the positive curvature operator $\Re$ of (*M, g*) is bounded from below then the sectional curvature of this manifold is positive and bounded from below too. On the other hand, we recall that a complete Riemannian manifold (*M, g*) is closed if its sectional curvature is bounded from below by some positive number (see: [4], p. 212—213).

Thus, using all these facts, we can formulate the following theorem.

Theorem 1. *Let* (*M′, g′*) *be an n′-dimensional simply connected and complete non-totally geodesic, totally umbilical submanifold in an n-dimensional Riemannian manifold* (*M*, *g*). *If the curvature operator* $\Re$ *of* (*M*, *g*) *is positive semi-definite on the bundle* $\Lambda^2 M'$ *over the submanifold* (*M′, g′*) *and the mean curvature* H^2 *of* (*M′, g′*) *reaches its lowest value* H^2_{min}, *then* (*M′, g′*) *is closed and homeomorphic to a sphere, its Betti numbers* $b_1(M')$, *…,* $b_{n'-1}(M')$ *are zero and an arbitrary conformal Killing r-form* ω *on* (*M′, g′*) *can be decomposed in the form* $\omega' + \omega''$ *where* ω' *is a Killing r-form and* ω'' *is a closed conformal Killing r-form for all r* = 1, …, $n' - 1$.

Proof. We consider a complete and simply connected non-totally geodesic, totally umbilical submanifold (*M′, g′*) of a Riemannian manifold (*M, g*). The tensor *R′* of (*M′, g′*) we can find from the Gauss curvature equation, which for a totally umbilical (*M′, g′*) in a Riemannian manifold (*M, g*) has the form

$$R'(X',Y',V',W') = R(f_*X', f_*Y', f_*V', f_*W') + \\ + H^2\big(g'(X',W')g'(Y',V') - g'(Y',W')g'(X',V')\big) \qquad (1)$$

for the *mean curvature* $H^2 = g(\mathcal{H}, \mathcal{H})$ of the submanifold (M', g') and any vector fields $X', Y', Z', W' \in C^\infty TM'$. We can rewritten the Gauss curvature equation (1) in the following form

$$g'(\Re'(\theta'), \theta') = g'(\Re(f^*\theta'), f^*\theta') + 2H^2 \|\theta'\|^2 \tag{2}$$

for an arbitrary $\theta' \in C^\infty \Lambda^2 M'$ and $\|\theta'\|^2 = g'(\theta', \theta')$.

Next, we suppose that there exists the positive number H^2_{min} which is the lowest value of the mean curvature H^2 of (M', g') and $g'(\Re(f^*\theta'), f^*\theta') \geq 0$ for an arbitrary $\theta' \in C^\infty \Lambda^2 M'$ then from (2) we obtain the inequalities

$$g'(\Re'(\theta'), \theta') \geq 2H^2_{\min} \|\theta'\|^2 > 0 . \tag{3}$$

In turn, from (3) we conclude that the curvature operator $\Re'$ of (M', g') is positive and bounded from below and hence the sectional curvature $sec'(\pi)$ of (M', g') is positive and bounded from below too. In this case, the simply connected complete manifold (M', g') is compact and homeomorphic to a sphere. Then its Betti numbers $b_1(M')$, …, $b_{n'-1}(M')$ are zero and an arbitrary conformal Killing r-form ω on (M', g') is uniquely decomposed in the form $\omega' + \omega''$ where ω' is a Killing r-form and ω'' is a closed conformal Killing r-form on (M', g') for all $r = 1, \ldots, n' - 1$ (see: [5]).

2. From the Gauss curvature equation (1) we can obtain identities relating sectional curvatures of the Riemannian manifold (M, g) and its totally umbilical submanifold (M', g') and the mean curvature of (M', g') (see also: [6])

$$\sec'(\pi) = \sec(\pi) + H^2 \tag{4}$$

where $\pi \subset T_x M'$ and $sec(\pi) = \dfrac{g(\Re(f_* X'_x \wedge f_* Y'_x), f_* X'_x \wedge f_* Y'_x)}{g(f_* X'_x \wedge f_* Y'_x, f_* X'_x \wedge f_* Y'_x)}$.

Schouten proved (see: [7], p. 301) that every totally umbilical submanifold of dimension ≥ 4 in conformally flat Riemannian manifold is conformally flat. For these manifolds the following proposition is true.

Corollary 2. *Let* (M', g') *be an* n'*-dimensional* $(n' \geq 4)$ *simply connected and complete non-totally geodesic, totally umbilical submanifold of an n-dimensional conformally flat Riemannian manifold* (M, g). *If the sectional curvature* $sec(\pi)$ *of* (M, g) *is positive semi-define for all two-plane section* π *of* TM' *and the mean curvature* H^2 *of* (M', g') *reaches its lowest value* H^2_{min}, *then* (M', g') *is conformally diffeomorphic to a sphere, its Betti number* $b_1(M'), \ldots, b_{n'-1}(M')$ *are zero and an arbitrary conformal Killing r-form* ω *can be decomposed in the form* $\omega' + \omega''$ *where* ω' *is a Killing r-form and* ω'' *is a closed conformal Killing r-form on* (M', g') *for all* $r = 1, \ldots, n'-1$.

Proof. Now, we consider a simply connected and complete non-totally geodesic, totally umbilical conformally flat submanifold (M', g') of a Riemannian manifold (M, g). Next, we suppose that there exists the positive number H^2_{min} which is the lowest value of the mean curvature H^2 of (M', g') and $sec(\pi) \geq 0$ for all two-plane section π of TM' then from (4) we obtain the inequalities $\sec'(\pi) = \sec(\pi) + H^2 \geq H^2_{min} > 0$. In this case, the simply connected complete manifold (M', g') is closed. In addition, we recall that a conformally flat simply connected closed Riemannian manifold (M', g') is conformally diffeomorphic to a sphere (see: [8]). In this case, Betti numbers $b_1(M'), \ldots, b_{n'-1}(M')$ of (M', g') are zero and an arbitrary conformal Killing r-form ω on (M', g') is uniquely decomposed in the form $\omega' + \omega''$ where ω' is a Killing r-form and ω'' is a closed conformal Killing r-form on (M', g') for all $r = 1, \ldots, n'-1$.

3. The *Ricci curvature* of (M', g') is

$$Ric(Y'_x, Z'_x) = trace\left(X'_x \to R(X'_x, Y'_x)Z'_x\right) = \sum_{i=1}^{n} g\left(R(e_i, Y'_x)Z'_x, e_i\right)$$

for an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ at an arbitrary point $x \in M'$. If $X'_x = e_1$, then $Ric(X'_x, X'_x) = \sum_{i=2}^{n} sec\left(X'_x, e_i\right)$. In dimension 3 we have the relation (see: [1, p. 38])

$$Ric'(e_1,e_1)=sec'(\pi_{12})+sec'(\pi_{13}); Ric'(e_2,e_2)=sec'(\pi_{12})+sec'(\pi_{23});$$
$$Ric'(e_3,e_3)=sec'(\pi_{23})+sec'(\pi_{13})$$

where $\pi_{12}=span\{e_1,e_2\}, \pi_{23}=span\{e_2,e_3\}$ and $\pi_{13}=span\{e_1,e_3\}$.

This means the sectional curvature $sec'(\pi)$ can be computed from *Ric'* and the sectional curvature determine Riemannian curvature operator $\mathfrak{R}'$. Moreover, if $\lambda_{12}(x)>\lambda_{23}(x)>\lambda_{13}(x)$ are eigenvalues of $\mathfrak{R}$ with respect to some orthonormal basis $\{e_1,e_2,e_3\}$ at an arbitrary point $x\in M'$ then we have (see: [1, p. 61])

$$\mathfrak{R}'_x=\begin{pmatrix}\lambda_{12}(x) & 0 & 0\\ 0 & \lambda_{23}(x) & 0\\ 0 & 0 & \lambda_{13}(x)\end{pmatrix},$$

$$Ric'_x=\frac{1}{2}\begin{pmatrix}\lambda_{23}(x)+\lambda_{13}(x) & 0 & 0\\ 0 & \lambda_{12}(x)+\lambda_{13}(x) & 0\\ 0 & 0 & \lambda_{12}(x)+\lambda_{23}(x)\end{pmatrix}.$$

Based on these facts and Theorem 1, we can prove the following corollary.

Corollary 3. *Let* (M', g') *be a three-dimensional simply connected and complete non-totally geodesic, totally umbilical submanifold of an n-dimensional Riemannian manifold* (M, g). *If the sectional curvature* $sec(\pi)$ *of* (M, g) *is positive semi-define for all two-plane section* π *of* TM' *and the mean curvature* H^2 *of* (M', g') *reaches its lowest value* H^2_{min}, *then* (M', g') *is closed and homeomorphic to a sphere, its Betti numbers* $b_1(M')$ *and* $b_2(M')$ *are zero and an arbitrary conformal Killing r-form* ω *can be decomposed in the form* $\omega' + \omega''$ *where* ω' *is a Killing r-form and* ω'' *is a closed conformal Killing r-form on* (M', g') *for* r = 1, 2.

Remark. If (M', g') is closed and $b_1(M')$ = 0 then any closed conformal Killing 1-form ω has the form $\omega = d\,f$ for a smooth scalar function f such that $\nabla d f = -n^{-1} \Delta f \cdot g$ where Δ is a Laplacian — Beltrami operator on (M', g'). In this case (M', g') is conformally diffeomorphic to a sphere (see: [10]). Therefore in every our proposition a simply connected and complete totally umbilical submanifold (M', g') is conformally diffeomorphic to a sphere if (M', g') admits a non-Killing conformal Killing 1-form.

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Теормы разложения конформно килинговых форм на тотально омбилических подмногообразиях

Доказываются теоремы о разложении конформно киллинговой r-формы в ортогональную сумму киллинговой и замкнутой конформно киллинговой r-форм на вполне омбилических поверхностях n-мерного риманова многообразия ($r = 1, \ldots, n - 1$).

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Инфинитезимальные преобразования аффинной связности касательного расслоения пространства нелинейной связности

В работе показано, что полный лифт X^C инфинитезимального преобразования X дифференцируемого многообразия M оставляет инвариантным аффинную связность касательного расслоения $T(M)$ пространства нелинейной связности тогда и только тогда, когда векторное поле X является инфинитезимальным движением в пространстве нелинейной связности.

Ключевые слова: касательное расслоение, нелинейная связность, полный лифт векторного поля, производная Ли, инфинитезимальное аффинное движение.

Пусть M — n-мерное дифференцируемое многообразие, $T(M)$ — касательное расслоение, π: $T(M) \to M$ — каноническая проекция, $G = R \setminus \{0\}$ группа Ли относительно операции умножения, действующая на касательном расслоении по закону: для любого $a \in G$ преобразование $R_a : T(M) \to T(M)$ отображает произвольный элемент $z \in T(M)$ в $R_a(z) = az$, где