A. K. Giżycki
Warsaw University of Technology, Poland
a.gizycki@mini.pw.edu.pl

Adapted topology in the sheaf of continuous functions

The book „Introduction to differential geometry" of Sikorski is about differential spaces. However, this book is about the sheaves. Indeed, the differential structure is a kind of sheaf. It seems, that theorems from this book can be proved for sheaves of continuous functions as for the differential spaces. We want to explain how the initial topology behaves when the sheaf of continuous functions is given. This topology adapts to the sheaf topology.

Key words: sheaves, topological spaces, initial topology.

1. Introduction

We recall the fundamentals of mathematics from the common books [1; 2].

Denotation. For any sets $X \neq \varnothing$ and $Y \neq \varnothing$ we denote: $Inj(X, Y)$ the set of all functions $f \in Y^X$ that are injections, $Sur(X, Y)$ the set of all functions $f \in Y^X$ that are surjections and $Bij(X, Y) = Inj(X, Y) \cap Sur(X, Y)$ the set of all bijections. For any topological space (M, τ) and for any $p \in M$ the family of neighbourhoods at p is $N_\tau(p) = \{U \in \tau | p \in U\} \subset \tau$.

Definition 1. For any set $M \neq \varnothing$, any set $S \subset 2^M$ and any set $C \subset R^M$

- $Top(M)=\{\varnothing\neq\tau\subset 2^M \mid \tau \text{ is a topology }\}\subset 2^{2^M}$ is a set of all topologies;
- $\Delta_S(M)=\{\tau\in Top(M) \mid S\subset\tau\}\subset Top(M)$ is a set of topologies that contains one set;
- $\Lambda_C(M)=\left\{\tau\in Top(M) \,\middle|\, \underset{f\in C}{\forall}\ \underset{a\in R}{\forall}\ f^{-1}((-\infty,a))\in\tau\right\}\subset Top(M)$ is a set of all topologies that has all functions continuous;
- $Inter(S)=\left\{A\subset M \,\middle|\, \underset{n\in N}{\exists}\ \underset{A_1,\ldots,A_n\in S}{\exists}\ A=\bigcap_{i=1}^{n}A_i\right\}\subset 2^M$ is a family of sets that are finite intersections from the set from one family;
- $\Gamma(S)=\{\varnothing,M\}\cup\left\{A\subset M \,\middle|\, \underset{B\in Inter(S)}{\exists}\ A=\bigcup B\right\}\subset 2^M$ is a set of all generalized sums of finite intersections of sets from one family;
- $\Xi(C)=\left\{A\subset M \,\middle|\, \underset{f\in C}{\exists}\ \underset{a\in R}{\exists}\ A=f^{-1}((-\infty,a))\right\}\subset 2^M$ is a family of preimages according of one function.

Recall [1] that induced topology is defined for any topological space (M,τ) and any $A\subset M$. This is the pair $\left(A,\tau|_A\right)$ where $\tau|_A=\left\{W\subset M \,\middle|\, \underset{U\in\tau}{\exists}\ W=U\cap A\right\}\subset 2^M$ and it is easy to see that it forms topology. In many booksthe concept of sheaf is explained in such a way that sheaf is a kind of presheaf that is a special functor between two categories. However we use here the equivalent definition from [3].

Definition 2. For any topological spaces (X,τ_X) and (S,τ_S)

- $C((S,\tau_S),(X,\tau_X))=\left\{f\in Y^X \,\middle|\, \underset{V\in\tau_X}{\forall}\ f^{-1}(V)\in\tau_S\right\}\subset Y^X$ is the set of continuous functions;

• $Homeo((S, \tau_S),(X,\tau_X))$ is the set of homeomorphisms i.e. the set of all functions $f \in Bij(S,X) \cap C((S,\tau_S),(X,\tau_X))$ such that $f^{-1} \in C((X,\tau_X),(S,\tau_S))$;

• $Homeo^{loc}((S,\tau_S),(X,\tau_X))$ is the set of local homeomorphisms i.e. the set of all functions $f \in X^S$ such that for any point s of S there is $U \in N_{\tau_S}(s)$ such that $f(U) \in \tau_X$ and $f|_U \in Homeo((U,\tau_S|_U),(f(U),\tau_X|_{f(U)}))$;

• $Homeo^{loc}_{sur}((S,\tau_S),(X,\tau_X))$ is the set of all surjective local homeomorphisms that we call sheaf projections;

• Triple $((S,\tau_S),\pi,(X,\tau_X))$ is a sheaf with sheaf space (S,τ_S) and base space (X, τ_X) if and only if $\pi \in Homeo^{loc}_{sur}((S,\tau_S),(X,\tau_X))$ ie. π is a sheaf projection.

If we consider all sheaves over given base (X, τ_X) we consider all topological spaces (S, τ_S) and all sheaf projections $\pi : S \to X$. Therefore we consider a class of sheaves that we denote as Sh_X. We recall definition of morphism of two sheaves.

Definition 3. For any topological space (X, τ_X), any sheaves $S_X = ((S,\tau_S),\pi,(X,\tau_X))$ and $T_X = ((T,\tau_T),\rho,(X,\tau_X))$ the set of morphisms between sheaves is the set of all continuous functions $\phi \in C((S,\tau_S),(T,\tau_T))$ such that $\rho \circ \varphi = \pi$. We denote this set as $Mor^{sh}(S_X,T_X)$.

If we consider the morphism between all sheaves over the base X we consider the class of functions. Class of all morphisms we denote as Mor_{Sh_X}. The book [2] mentions that all sheaves over given base form a category of sheaves with composition of morphisms of sheaves and identity morphism in the same manner as in category of sets.

2. Initial topology and differential spaces

Initial topology is well known object in Sikorski differential spaces. However there is no book with full formal proofthat shows what is in fact initial topology. Usually it is left as an excercise to proof that initial topology is well defined. Here we are going to explain in details this proof.

Lemma. For any set M, for any set $S \subset 2^M$ and for any set $C \subset R^M$ the following facts holds

- $\Delta_S(M) \neq \varnothing$ and $\Lambda_C(M) \neq \varnothing$;
- $D_S(M) = \bigcap \Delta_S(M)$ and $\tau_C(M) = \bigcap \Lambda_C(M)$ are well defined;
- $\tau_C(M) \in \Lambda_C(M)$ and $D_S(M) \in \Delta_S(M)$;
- $\underset{\tau \in \Delta_S(M)}{\forall} D_S(M) \subset \tau$ and $\underset{\tau \in \Lambda_C(M)}{\forall} \tau_C(M) \subset \tau$;
- $\Gamma(S) \in \Delta_S(M)$ and $\Gamma(S) = D_S(M)$;
- $\tau_C(M) = D_{\Xi(C)}(M)$.

Proof. Notice that $S \subset 2^M$ and $2^M \in \Delta_S(M)$ and therefore $\Delta_S(M) \neq \varnothing$. Similarly we have that $\underset{f \in C}{\forall} \underset{a \in R}{\forall} f^{-1}((-\infty, a)) \in 2^M$ and $2^M \in \Lambda_C(M)$ and therefore $\Lambda_C(M) \neq \varnothing$. This is the reason why $\bigcap \Delta_S(M)$ and $\bigcap \Lambda_C(M)$ are well defined. Then for any $\tau \in \Delta_S(M)$ we have that $\bigcap \Delta_S(M) \subset \tau$ and for any $\tau \in \Lambda_C(M)$ we have that $\bigcap \Lambda_C(M) \subset \tau$. We know from the basics of topology that for any $M \neq \varnothing$ and any family of topological spaces $A \subset Top(M)$ the intersection is a topology $\bigcap A \in Top(M)$. Thus we have two topologies $\bigcap \Delta_S(M) \in Top(M)$ and $\bigcap \Lambda_C(M) \in Top(M)$. To show further facts assume that for any $A \subset M$ we have $A \in S$. Notice that for any $\tau \in \Delta_S(M)$ we have that $S \subset \tau$ and therefore

$\underset{\tau \in \Delta_S(M)}{\forall} A \in \tau$. This means that $A \in \bigcap \Delta_S(M) = D_S(M)$. Notice that $\bigcap \Delta_S(M) \in \Delta_S(M)$. It is easy to see that for any topology $\tau \in \Lambda_C(M)$, any function $f \in C$ and any number $a \in R$ we have that $f^{-1}((-\infty, a)) \in \tau$. Reversing the quantifiers we obtain that for any function $f \in C$, for any $a \in R$ and for any topology $\tau \in \Lambda_C(M)$ we have that $f^{-1}((-\infty, a)) \in \tau$.

Therefore $\underset{\tau \in \Lambda_C(M)}{\forall} f^{-1}((-\infty, a)) \in \tau$ but this means that $f^{-1}((-\infty, a)) \in \bigcap \Lambda_C(M)$ for any function $f \in C$ and any numer $a \in R$. Notice that $\bigcap \Lambda_C(M) \in \Lambda_C(M)$. Straight from the definition we have that $\varnothing, M \in \Gamma(S)$. If we have $A, B \in \Gamma(S)$ then there are subfamilies $K, L \subset \mathit{Inter}(S)$ such that $A = \bigcup K$ and $B = \bigcup L$. Therefore A and B are sums of finite intersections from the family S. Then $A \cap B$ is the sum of finite intersections fromfamily S. Therefore $A \cap B = \bigcup P$ where subfamily $P \subset \mathit{Inter}(S)$, so $A \cap B \in \Gamma(S)$. If $A \subset \Gamma(S)$ then A is a family of sums of finite intersections from S. Therefore $\bigcup A$ is a sum of finite intersections fromfamily S, so $\bigcup A \in \Gamma(S)$. Getting all together we see that the family $\Gamma(S) \subset 2^M$ satisfies $\Gamma(S) \in \mathit{Top}(M)$ and for any set $A \in S$ the set A is one element sum of the set $A \cap M$. Therefore $A \in \Gamma(S)$ and $S \subset \Gamma(S)$. Getting all together we see that $\Gamma(S) \in \Delta_S(M)$.

Therefore $\bigcap \Delta_S(M) = D_S(M) \subset \Gamma(S)$. Now we show the implication in other direction. Notice that $D_S(M) \in \Delta_S(M)$ and we have that $\varnothing, M \in D_S(M)$. From the previous consideration we have that $S \subset D_S(M)$. This gives us that for any $n \in N$ and any $U_1, ..., U_n \in S$ we have that $U_1 \cap ... \cap U_n \in D_S(M)$. Notice that

$D_S(M) \in \Delta_S(M)$. This gives us that all sums of finite intersections are in $D_S(M)$. Therefore $\Gamma(S) \subset D_S(M)$. Getting all together we see that $\Gamma(S) = D_S(M)$. For any $A \subset M$ assume that $A \in \Xi(C)$. Then there are $f \in C$ and $a \in R$ such that $A = f^{-1}((-\infty, a))$. Notice that for any $\tau \in \Lambda_C(M)$ we get from definition that $A = f^{-1}((-\infty, a)) \in \tau$. Therefore $A \in \bigcap \Lambda_C(M) = \tau_C(M)$.

Getting all together we see that $\Xi(C) \subset \tau_C(M)$. Notice that $\tau_C(M) \in Top(M)$. The fact that $\Xi(C) \subset \tau_C(M)$ gives us that $\tau_C(M) \in \Delta_{\Xi(C)}(M)$ and therefore $D_{\Xi(C)}(M) = \bigcap \Delta_{\Xi(C)}(M) \subset$ $\subset \tau_C(M)$. To show the opposite inclusion notice that for any $f \in C$ and any $a \in R$ we have from definition that $f^{-1}((-\infty, a)) \in \Xi(C)$. Notice that $\Xi(C) \subset D_{\Xi(C)}(M)$ and therefore $\underset{f \in C}{\forall} \underset{a \in R}{\forall} f^{-1}((-\infty, a)) \in D_{\Xi(C)}(M)$. Notice that $D_{\Xi(C)}(M) \in Top(M)$ and therefore $D_{\Xi(C)}(M) \in \Lambda_C(M)$. This is the reason why we obtain the inclusion $\tau_C(M) = \bigcap \Lambda_C(M) \subset D_{\Xi(C)}(M)$ and getting all togetherwe see that $D_{\Xi(C)}(M) = \tau_C(M)$. If this can't lead to misunderstanding the letter M is omitted and we write $\tau_C = \tau_C(M)$.

Recall the basic definitions from the book [3].

Definition 4. For anyset M, any family of functions $C \subset R^M$, any point $p \in M$, and any set $A \subset M$

- $C_p = \left\{ f \in R^M \,\middle|\, \underset{U \in N_{\tau_C(M)}(p)}{\exists} \underset{g \in C}{\exists} g|_U = f|_U \right\} \subset R^M$;
- $C|_A = \{ f|_A \mid f \in C \}$;
- $C_M = \bigcap\limits_{p \in M} C_p$;

- $sc(C)=\left\{ f \in R^M \,\middle|\, \underset{n\in N}{\exists}\ \underset{f_1,\dots,f_n\in C}{\exists}\ \underset{\omega\in C^\infty\left(R^n,R\right)}{\exists} f=\omega\circ\left(f_1,\dots,f_n\right)\right\}$.

Definition 5. For any set M and any $C \subset R^M$ the set C is differential structure if and only if $C = C_M = sc(C)$. Then the pair (M, C) is called differential space.

3. Sheaf of continuous functions and adapted topology

According to [2] we consider the sheaf of continuous functions $(C(U))_{U\in\tau}$ in topological space (X,τ). Denote τ_C the weakest topology such that all function from the sheaf of continuous functions are continuous. This requirement means that for any $f \in C$ and for any $a \in R$ we have that $f^{-1}((-\infty, a)) \in \tau_C\big|_{dom(f)}$. We will show that the fact of choice a sheaf of continuous functions ensures us thatthe initial topology fits to this sheaf. The behavior of initialtopology is adaptation. In this situation we can work with sheaves of continuous functions in a similar manner to differential spaces.

Theorem. *For any sheaf of continuous functions* $\tau = \tau_C$.

Proof. Topology τ_C is the weakest such that all functions f from the sheaf of continuous functions are continuous. For any function $f : dom(f) = U \to R$ this function is continuous in induced topological space $\left(U, \tau|_U\right)$, so $\tau_C \subset \tau$. The question is whether there is inclusion in the opposite direction $\tau \subset \tau_C$. Assume that we have any $U \in \tau$. Then we have the family of continuous functions $C(U, R)$. Assume additionally that all constant functions are in $C(U, R)$. It turns out that for any sheaf of continuous functions all constant functions are in $C(U, R)$. Indeed, any sheaf that is closed due to superposition has constant functions in all its $C(U,R)$. Notice that there is no algebraic sheaf such that $C(U,R)=\varnothing$ and therefore every algebraic sheaf is closed due to superposition. Let $P=\prod_{i=1}^{n}(a_i,b_i)$ be an open interval in R^n. Notice that for any $n\in N$ and any constant functions $f_1,\dots,f_n$ we obtain that $(f_1,\dots,f_n)^{-1}(P)\in\tau|_U$. Therefore for any $p\in U$ there are constant functions $(f_1,\dots,f_n):U\to R^n$ such that $W=(f_1,\dots,f_n)^{-1}(P)\in\tau_C$. Indeed, the initial topology τ_C has a base consisting of the pre-images of all intervals. Notice that U is the sum of those $W\in\tau_C$ as the aforementioned. But each sum of elements of topology τ_C belongs to τ_C, so $U\in\tau_C$.

Bibliography

1. *Engelking R.* General Topology // PWN. Warsaw, 1989.
2. *Mallios A.* Geometry of Vector Sheaves : An Axiomatic Approach to Differential Geometry. Springer, 1998.
3. *Sikorski R.* Introduction to differential geometry // PWN. Warsaw, 1972.

А. К. Гижицкий

Адаптированная топология в пучке непрерывных функций

Книга Р. Сикорского «Введение в дифференциальную геометрию» посвящена дифференцируемым пространствам. Однако эта книга — о пучках. Действительно, дифференцируемая структура — это в определенном смысле пучок. Похоже, что теоремы этой книги могут быть доказаны для пучков непрерывных функций так же, как это сделано для дифференцируемых пространств. Мы хотим объяснить, как инициальная топология ведет себя в случае пучка непрерывных функций: эта топология адаптируется топологии пучка.