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Complete Riemannian manifolds with Killing — Ricci and Codazzi — Ricci tensors

The purpose of this paper is to prove of Liouville type theorems, i. e., theorems on the non-existence of Killing — Ricci and Codazzi — Ricci tensors on complete non-compact Riemannian manifolds. Our results complement the two classical vanishing theorems from the last chapter of famous Besse's monograph on Einstein manifolds.

Keywords: complete Riemannian manifold, Killing — Ricci tensor, Codazzi — Ricci tensor, Liouville-type theorems

1. Introduction

A. Gray introduced in [1] two classes of Riemannian manifolds $\mathcal{A}$ and $\mathcal{B}$, which are defined by the two following conditions on the covariant derivative of the Ricci tensor. Firstly, a Riemannian manifold (M, g) belongs to $\mathcal{A}$ if and only if its Ricci tensor Ric is a *Killing tensor*, that is,

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(X, Z) + (\nabla_Z Ric)(X, Y) = 0 \qquad (1.1)$$

for all $X, Y, Z \in TM$. In this case, Ric is called the *Killing — Ricci tensor* (see [2]). Second, a Riemannian manifold (M, g)

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belongs to $\mathcal{B}$ if and only if its Ricci tensor Ric is a *Codazzi tensor*, that is,

$$(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) = 0 \qquad (1.2)$$

for all $X, Y, Z \in TM$. In this case, Ric is called the *Codazzi — Ricci tensor* (see [3]).

Obviously, all manifolds belonging to $\mathcal{A}$ or $\mathcal{B}$, which are known as *Einstein-like manifolds*, have constant scalar curvature $s = \mathrm{trace}_g Ric$. Moreover, any manifold that belongs to $\mathcal{A} \cap \mathcal{B}$ must have a parallel Ricci tensor. An example of this type of Einstein-like manifolds is a *Riemannian locally symmetric space* (see [4, p. 369]). More interesting examples which are Einstein-like but not Einstein can be found in [5, p. 432—455].

The aim of this paper is to prove Liouville-type theorems, i. e., non-existence theorems for complete noncompact manifolds of classes $\mathcal{A}$ and $\mathcal{B}$. Our results complement two classical theorems of the last chapter of Besse's famous monograph [7].

2. Liouville-type theorems for complete Einstein-like manifolds of class $\mathcal{A}$

Let (M, g) be a Riemannian Einstein-like manifold (M, g) of class $\mathcal{A}$. Then its Ricci tensor Ric satisfies the equations (1.1) and has a constant trace, i. e., the scalar curvature $s = \mathrm{trace}_g Ric$ is a constant function. This also means that the Ricci tensor is a divergence-free tensor.

It is known that if (M, g) is a compact (without boundary) Einstein-like manifolds of class $\mathcal{A}$ with non-positive sectional curvature, then $\nabla Ric = 0$. If in addition there exists a point in M where the sectional curvature of every two-plane is strictly negative, then (M, g) is Einstein, i. e., its Ricci tensor satisfies $Ric = \rho g$ for some constant ρ (see [5, p. 451]).

On the other hand, from [6] we conclude that the following theorem holds: On a simply connected complete Riemannian manifold (M, g) of nonpositive sectional curvature, any divergence-free Killing 2-tensor, such that $\|\varphi\| \in L^p$ for at least one $p \in (0, \infty +)$, is a parallel tensor field. If, in addition, the volume of the manifold is infinite, then there exist no nonzero divergence-free Killing 2-tensors. In turn, we recall here that a simply connected complete Riemannian manifold (M, g) of nonpositive curvature is called a *Hadamard manifold* after the Cartan — Hadamard theorem (see, for example, [4, p. 241]). From the Cartan — Hadamard theorem one can conclude, in particular, that no compact simply connected manifold admits a metric of nonpositive curvature (see also [4, p. 162]). Moreover, Hadamard manifolds have infinite volume (see [8]). Therefore, the Ricci tensor of a Hadamard manifold, which is a Riemannian Einstein-like manifold (M, g) of class $\mathcal{A}$, is equals to zero. In this case, the sectional curvature must vanishes in (M, g). Then (M, g) is a flat manifold. Again (M, g) is a simply connected manifold, hence it follows that (M, g) is isometric to the Euclidean space $\mathbf{R}^n$.

Theorem 1. *Let an n-dimensional Riemannian Einstein-like manifold* (M, g) *of class* $\mathcal{A}$ *be a Hadamard manifold. If* $\|\varphi\| \in L^p$ *for at least one* $p \in (0, +\infty)$*, then* (M, g) *is isometric to the Euclidean space* $\mathbf{R}^n$.

2. Liouville-type theorems for complete Einstein-like manifolds of class $\mathcal{B}$

Let (M, g) be a Riemannian Einstein-like manifold (M, g) of class $\mathcal{B}$. Then its Ricci tensor Ric satisfies the equations (1.2) and has a constant trace, i. e. the scalar curvature $s = \mathrm{trace}_g Ric$ is a constant function. This also means that the Ricci tensor is a divergence-free tensor. In this case, Ric is a *symmetric harmonic* 2-*tensor* (see [4, p. 350]).

The following classical Berger — Ebin theorem is well known: If (M, g) is a compact (without boundary) Einstein-like manifolds of class $\mathcal{B}$ with non-negative sectional curvature, then $\nabla Ric = 0$. If in addition there exists a point in M where the sectional curvature of every two-plane is strictly positive, then (M, g) is Einstein (see [7, p. 445]).

On the other hand, from [9] we conclude the following theorem: Let (M, g) be a connected complete noncompact Riemannian manifold with nonnegative sectional curvature. Then there is no a non-zero harmonic symmetric 2-tensor φ which satisfies the condition $\|\varphi\| \in L^p$ for at least one $p \in (1, +\infty)$. Therefore, the Ricci tensor of a connected complete noncompact Riemannian manifold with nonnegative sectional curvature (M, g) of class $\mathcal{B}$ is equals to zero. In this case, the sectional curvature must vanishes in (M, g). Then (M, g) is a flat manifold. Again if (M, g) is a simply connected manifold, hence it follows that (M, g) is isometric to the Euclidean space $\mathbf{R}^n$. Therefore, we can formulate a theorem.

Theorem 2. *Let a Riemannian Einstein-like manifold (M, g) of class $\mathcal{B}$ be a connected complete noncompact Riemannian manifold with nonnegative sectional curvature. If $\|\varphi\| \in L^p$ for at least one $p \in (1, +\infty)$, then (M, g) is a flat manifold. If, moreover, (M, g) is a simply connected manifold, then (M, g) is isometric to the Euclidean space $\mathbf{R}^n$.*

References

1. *Gray, A.:* Einstein-like manifolds which are not Einstein. Geom. Dedicata, **7**:3, 259—280 (1978).

2. *Suh, Y.J.:* Generalized Killing Ricci tensor for real hypersurfaces in complex two-plane Grassmannians. J. Geom. Phys., 159, 103799 (2011). doi: 10.1016/j.geomphys.2020.103799.

3. *Calvaruso, G.:* Riemannian 3-metrics with a generic Codazzi Ricci tensor. Geom. Dedicata, **151**:1, 259—267 (2011).

4. *Petersen, P.:* Riemannian Geometry. Springer (2016).

5. *Besse, A.L.:* Einstein Manifolds. Berlin Heidelberg (2008).

6. *Stepanov, S.E.; Tsyganok, I.I.:* Codazzi and Killing tensors on a complete Riemannian manifold. Math. Notes, **109**:6, 901—911 (2021). (In Rus.).

7. *Besse, A.L.:* Einstein Manifolds. Springer (1987).

8. *Bertrand, J., Sandeep, K.:* Sharp Green's function estimates on Hadamard manifolds and Adams inequality. International Mathematics Research Notices, 6, 4729—4767 (2021).

9. *Mikes, J., Rovenski, V., Stepanov, S.:* On higher order Codazzi tensors on complete Riemannian manifolds. Annals of Global Analysis and Geometry, 56, 429—442 (2021).

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Полные многообразия с тензорами Киллинга — Риччи и Кодацци — Риччи

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Целью работы является доказательство теорем лиувиллева типа, то есть теорем несуществования для тензоров Киллинга — Риччи и Кодацци — Риччи на полном некомпактном римановом многообразии. Наши результаты дополняют две классические теоремы исчезновения из последней главы известной монографии А. Бессе.

Ключевые слова: полное риманово многообразие, тензор Киллинга — Риччи, тензор Кодацци — Риччи, теоремы Лиувиллева типа

Список литературы

1. *Gray A.* Einstein-like manifolds which are not Einstein // Geom. Dedicata. 1978. Vol. 7, №3. P. 259—280.

2. *Suh Y.J.* Generalized Killing Ricci tensor for real hypersurfaces in complex two-plane Grassmannians // J. Geom. Phys. 2011. №159. Art. №103799. doi: 10.1016/j.geomphys.2020.103799.

3. *Calvaruso G.* Riemannian 3-metrics with a generic Codazzi Ricci tensor // Geom. Dedicata. 2011. Vol. 151, №1. P. 259—267.

4. *Petersen P.* Riemannian Geometry. Springer, 2016.

5. *Besse A.L.* Einstein Manifolds. Berlin ; Heidelberg, 2008.

6. *Степанов С.Е., Цыганок И.И.* О тензорах Кодацци и Киллинга на полном римановом многообразии // Матем. заметки. 2021. Т. 109, вып. 6. С. 901—911.

7. *Besse A.L.* Einstein Manifolds. Springer, 1987.

8. *Bertrand J., Sandeep K.* Sharp Green's function estimates on Hadamard manifolds and Adams inequality // International Mathematics Research Notices. 2021. Vol. 6. P. 4729—4767.

9. *Mikes J., Rovenski V., Stepanov S.* On higher order Codazzi tensors on complete Riemannian manifolds // Annals of Global Analysis and Geometry. 2021. Vol. 56. P. 429—442.