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Some remarks on 2-Killing vector fields

The aim of this paper is to comment on 2-Killing vector fields which are an extension of Killing vector fields. Some algebraic structures of these vector fields are studied and various comparisons are made with Killing vector fields which are fundamental in the study of symmetries. A number of evaluations are conducted and several remarks are issued related to these vector fields. The existence of 2-Killing vector fields on specific 4-dimensional manifolds is investigated and some examples are constructed in this manner. More specifically, vector fields that are 2-Killing but not Killing are established. Furthermore, it is proven that the set of 2-Killing vector fields does not satisfy some basic properties, such as forming a vector space or a Lie algebra and curvature collineation.

Keywords: 2-Killing vector field, Killing vector field, Lie algebra, 4-dimensional manifolds

1. Introduction

The notion of symmetry is an essential concept both in differential geometry and physics, being a transformation under which certain geometric or physical properties unchanged. In other words, a symmetry of a geometric structure on a (smooth) Riemannian or semi-Riemannian manifold is a diffeomorphism preserving some geometric data. Such a transformation is usually expressed by us-

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ing Lie derivatives, and we often attempt to understand a symmetry through infinitesimally transformations. These infinitesimal transformations are defined by vector fields on a manifold, some of the well-known ones being affine, conformal, Killing, homothetic and projective vector fields (see, e. g., [9; 10]). Among these, Killing vector fields are very important in terms of reflecting the symmetry of the metric. Given a smooth manifold M with a metric g, a Killing vector field, say V, is a smooth vector field leading to a transformation that leaves g unchanged, and which satisfies the condition $\mathcal{L}_V g = 0$, where $\mathcal{L}_V$ represents the Lie derivative along V. To put it another way, one has $\varphi_t^*(g) = g$ for the pullback φ_t^* of any local diffeomorphism φ_t arising from V. Notably, the collection of all such vector fields is a Lie algebra, referred to as the *Killing algebra*, equipped with commutator [,], of finite dimension, indicated by $\mathcal{K}(M)$. For $V \in \mathcal{K}(M)$, the following Killing equations are satisfied:

$$\nabla_j V_i + \nabla_i V_j = 0 \ \left(\Leftrightarrow \nabla_j V_i = F_{ij} = -F_{ji}\right),$$
$$\nabla_k \nabla_j V^i = \nabla_k F^i{}_j = \mathcal{R}^i{}_{jkl} V^l, \qquad (1)$$

where V^i and $\mathcal{R}^i{}_{jkl}$ are the components of V and the Riemann tensor, respectively. Furthermore, the 2-form F in (1) is called the *Killing bivector* of V with components $F_{ij} = -F_{ji}$. For a smooth manifold M of dimension n, note that

$$\dim\mathcal{K}(M) \leq \frac{1}{2} n(n+1)$$

(for details, see, e. g., [9; 12]).

The concept of *2-Killing vector fields* is inspired by, and emerges as a generalization of, Killing vector fields which are significant both mathematically and physically (see [13]). A 2-Killing vector field on M is a vector field V satisfying

$$\mathcal{L}_V \mathcal{L}_V g = 0, \qquad (2)$$

that is, the second Lie derivative of g in the direction of V vanishes. Each $V \in \mathcal{K}(M)$ is necessarily a 2-Killing vector field, but the

converse does not hold in general and for which counter examples are provided in Section 3. Recently, there has been growing interest in 2-Killing vector fields; among these, studies concerning warped, doubly warped, multiply warped product manifolds and multiply twisted product space-times can be found respectively in [14], [1], [3] and [2]. Additionally, more recent papers have analyzed these vector fields in almost contact metric structures [4; 8], hyperbolic Yamabe solitons [5], Poisson doubly warped product manifolds [6], and sequential doubly warped products [7].

As outlined above, while 2-Killing vector fields have been studied with rising curiosity in prior research, some of their essential properties do not seem to have been presented explicitly. This paper attempts to make these main features more precise and to highlight the algebraic differences between Killing and 2-Killing vector fields. These points are supported by illustrative examples. Within this framework, Section 2 provides a comparison between Killing and 2-Killing vector fields. Demonstrations of these discussions are presented in Section 3.

2. Some comparisons of Killing and 2-Killing vector fields

Building on the discussion of Section 1, the algebraic structure of 2-Killing vector fields will now be discussed in more detail. Let $\mathcal{K}'(M)$ denote the set of 2-Killing vector fields. For $V \in \mathcal{K}'(M)$, the local expression of (2) is written as

$$\begin{aligned} \mathcal{L}_V \mathcal{L}_V g = \mathcal{L}_V\big(\nabla_j V_i + \nabla_i V_j\big) = \\ = V^k \nabla_k\big(\nabla_j V_i + \nabla_i V_j\big) + \big(\nabla_i V^k\big)\big(\nabla_k V_j + \nabla_j V_k\big) + \\ + \big(\nabla_j V^k\big)(\nabla_k V_i + \nabla_i V_k) = 0, \end{aligned} \tag{3}$$

where ∇ is the Levi-Civita connection of g. Clearly, a member of $\mathcal{K}(M)$ satisfies the equation (3), whereas Killing equations given by (1) do not necessarily apply to a member of $\mathcal{K}'(M)$. Accordingly, the following remarks are also worth mentioning:

Remark 1. $\mathcal{K}'(M)$ generally does not constitute a vector space. For example, $\mathcal{K}'(M)$ is not closed under addition. In a clearer way, let V and W be members of $\mathcal{K}'(M)$. By using (2), one has

$$\mathcal{L}_{V+W}\left(\mathcal{L}_{(V+W)}g\right) =$$
$$= \mathcal{L}_V\mathcal{L}_V g + \mathcal{L}_V\mathcal{L}_W g + \mathcal{L}_W\mathcal{L}_V g + \mathcal{L}_W\mathcal{L}_W g =$$
$$= \mathcal{L}_V\mathcal{L}_W g + \mathcal{L}_W\mathcal{L}_V g. \qquad (4)$$

It is evident that Eq. (4) does not have to equal zero and the anticipated result is obtained. Even if one of them were an element of $\mathcal{K}(M)$, it would not be sufficient to satisfy the requirement (see Section 3).

Remark 2. Once again, it is important to note that $\mathcal{K}(M)$ is a Lie algebra. For $V, W \in \mathcal{K}(M)$, one has $\mathcal{L}_{[V,W]}g = 0$, and so $[V, W] \in \mathcal{K}(M)$. But, $\mathcal{K}'(M)$ generally fails to produce a Lie algebra structure since it is not a vector space. In this respect, $\mathcal{K}'(M)$ differs significantly from $\mathcal{K}(M)$.

Remark 3. While Killing symmetries preserve the Riemann tensor, denoted by $\mathcal{R}$, members of $\mathcal{K}'(M)$ often do not. As is well known, if $V \in \mathcal{K}(M)$, it is a curvature collineation, i. e., $\mathcal{L}_V\mathcal{R} = 0$. However, this is not valid for a member of $\mathcal{K}'(M)$ as demonstrated by a counterexample in Section 3.

Remarks 1—3 imply that although $\mathcal{K}(M)$ and $\mathcal{K}'(M)$ are defined by similar geometric objects and tools, they are in fact algebraically different. Thus, in these aspects, 2-Killing vector fields are distinct from well-known symmetries such as affine vector fields and their subclasses.

3. Existence of 2-Killing vector fields on some special space-times

Here, we provide illustrative examples to clarify the remarks made in Section 2. Some of the calculations were performed using Maple software.

Example 1. On $M = \mathbb{R}^4$ with coordinates (u, v, x, y), take the metric via the line element

$$ds^2 = y(u^2 + x^2)du^2 + 2dudv + dx^2 + dy^2, \qquad (5)$$

where $y > 0$. When the coefficient of du^2 in (5) is generally a smooth function such as $H(u, x, y)$, then (M, g) represents a *pp-wave space-time*, a significant concept in general relativity (see, e. g., [9; 15]). The metric (5) admits precisely one (independent) Killing vector field $l \coloneqq \frac{\partial}{\partial v}$, and so $\dim\mathcal{K}(M) = 1$. Define a vector field $V = y\frac{\partial}{\partial v} + \frac{\partial}{\partial y}$ on M. In this case, one gets

$$\mathcal{L}_V g = (u^2 + x^2)dudu + dudy + dydu. \qquad (6)$$

Clearly, $\mathcal{L}_V g \neq 0$, and so $V \notin \mathcal{K}(M)$. However, by taking the Lie derivative of (6) with respect to V once again, we get $\mathcal{L}_V\mathcal{L}_V g = 0$, which proves that V is a 2-Killing vector field on M.

On the other hand, the components of $\mathcal{R}$ that are non-zero are calculated by

$$\mathcal{R}^v_{xuy} = \mathcal{R}^x_{uyu} = \mathcal{R}^y_{uxu} = \mathcal{R}^v_{yux} = -\mathcal{R}^v_{yxu} = -\mathcal{R}^v_{xyu} =$$
$$= -\mathcal{R}^y_{uux} = -\mathcal{R}^x_{uuy} = -x,$$
$$\mathcal{R}^v_{xux} = \mathcal{R}^x_{uxu} = -\mathcal{R}^v_{xxu} = -\mathcal{R}^x_{uux} = -y,$$

which yield

$$(\mathcal{L}_V\mathcal{R})^v_{xux} = (\mathcal{L}_V\mathcal{R})^x_{uxu} = -(\mathcal{L}_V\mathcal{R})^v_{xxu} = -(\mathcal{L}_V\mathcal{R})^x_{uux} = -1,$$
$$(\mathcal{L}_V\mathcal{R})^v_{uux} = -(\mathcal{L}_V\mathcal{R})^v_{uxu} = -x$$

meaning that $\mathcal{L}_V\mathcal{R} \neq 0$, and so the situation described in the Remark 3 occurs here.

On top of that, if we define vector fields $U \coloneqq xy\frac{\partial}{\partial v}$ and $W \coloneqq \frac{\partial}{\partial y}$, it turns out that

$$\mathcal{L}_U g = x(dudy + dydu) + y(dudx + dxdu), \qquad (7)$$

and

$$\mathcal{L}_W g = (u^2 + x^2)dudu. \qquad (8)$$

Taking the Lie derivative once again over U and W respectively in equations (7) and (8), we find

$$\mathcal{L}_U\mathcal{L}_U g = 0 = \mathcal{L}_W\mathcal{L}_W g,$$

which guarantees that both U and W are members of $\mathcal{K}'(M)$. In spite of that, the sum of U and W gives

$$\mathcal{L}_{U+W}g = (u^2 + x^2)dudu + x(dudy + dydu) + \\ + y(dudx + dxdu),$$

and

$$\mathcal{L}_{U+W}\big(\mathcal{L}_{(U+W)}g\big) = dudx + dxdu. \qquad (9)$$

Thus, $U + W \notin \mathcal{K}'(M)$ by equation (9), and this serves as an example of the situation mentioned in Remarks 1 and 2. This leads to the following result:

Theorem 1. *The space-time endowed with the metric (5) contains* 2*-Killing vector fields* $y\frac{\partial}{\partial v} + \frac{\partial}{\partial y}$, $xy\frac{\partial}{\partial v}$, $\frac{\partial}{\partial y}$ *which are not Killing, and these lead to the conclusion that* $\mathcal{K}'(M)$ *does not form a Lie algebra structure.*

Example 2. Now, let us replace the coefficient of du^2 in the metric (5) by $u(u^2 + x^2)$, where $u > 0$. For this case, $X \coloneqq \\ \coloneqq -y\frac{\partial}{\partial v} + u\frac{\partial}{\partial y}$ is a member of $\mathcal{K}(M)$, i. e., $\mathcal{L}_X g = 0$. Next, consider the vector field $Y \coloneqq y\frac{\partial}{\partial v}$. Then, one gets

$$\mathcal{L}_Y g = dudy + dydu \ \text{ and } \ \mathcal{L}_Y\mathcal{L}_Y g = 0,$$

which tells us that $Y \in \mathcal{K}'(M)$. Nevertheless, it can be computed that $\mathcal{L}_{X+Y}\big(\mathcal{L}_{(X+Y)}g\big) = 2dudu$, that is, $X + Y \notin \mathcal{K}'(M)$, and hence the argument discussed in Remark 1 is observed here.

Example 3. In local coordinates (u, v, x, y) on $M = \mathbb{R}^4$, consider the metric with the line element

$$ds^2 = (u^2 - v^2)du^2 + 2dudv + dx^2 + dy^2. \qquad (10)$$

Let us begin by establishing that the metric (10) admits a Ricci soliton arising as special solutions to the Ricci flow introduced by R. S. Hamilton [11], which is described by the equation

$$\frac{1}{2}\mathcal{L}_U g + Ric = \lambda g, \tag{11}$$

where λ denotes a constant and Ric is the Ricci tensor. Firstly, for the metric (10), one obtains

$$Ric = -(u^2 - v^2)dudu - 2dudv. \tag{12}$$

Define $U = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. In this case, one gets

$$\mathcal{L}_U g = -2dxdx - 2dydy,$$

which yields together with equation (12)

$$\frac{1}{2}\mathcal{L}_U g + Ric + g = 0. \tag{13}$$

Comparing equations (11) and (13) results that the structure (M, g, U, λ) is a model of an *expanding Ricci soliton* for $\lambda = -1 < 0$ and the potential field U.

Next, let $V = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$, in which case we obtain

$$\mathcal{L}_V g = (2u - 2v)dudu,$$

and this gives rise to $\mathcal{L}_V \mathcal{L}_V g = 0$. Accordingly, $V \in \mathcal{K}'(M)$ and one can check that the potential field U is orthogonal to V. From these, the following theorem follows:

Theorem 2. *The space-time endowed with the metric (10) is a model of an expanding Ricci soliton for $\lambda = -1 < 0$, having the potential field $U = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. Moreover, it contains a 2-Killing vector field $V = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ which is orthogonal to U.*

Remark 4. Beyond what has been said above, a much more generalized condition for Killing vector fields can also be defined, giving rise to the concept of "n-Killing vector fields". To give an example, let us look at the metric (10) again. We define the vector field W as $u\frac{\partial}{\partial x}$. In this case, we get the following relations:

$$\mathcal{L}_W g = dudx + dxdu, \qquad \mathcal{L}_W \mathcal{L}_W g = 2dudu,$$
$$\mathcal{L}_W \mathcal{L}_W \mathcal{L}_W g = 0. \tag{14}$$

Equations (14) state that W is neither Killing nor 2-Killing. However, it can be considered as a vector field called "3-Killing" since the third order Lie derivative of g with respect to W vanishes.

4. Conclusion

This study investigates several basic properties of 2-Killing vector fields, considered as a generalization of Killing vector fields, and constructs examples of them on 4-dimensional manifolds, particularly on space-times. As can be observed with examples, the set of 2-Killing vector fields (i. e., $\mathcal{K}'(M)$) is algebraically quite different from the set of Killing vector fields (i. e., $\mathcal{K}(M)$), as it cannot even form the much more fundamental vector space structure. Therefore, the author would like to emphasize that 2-Killing vector fields do not reflect as specific and nice symmetry properties as Killing vector fields. Already highlighted, a 2-Killing vector field V does not have to preserve the metric g directly. But, it maintains the rate of change of g along the flow generated by V.

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Некоторые замечания о 2-векторных полях Киллинга

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Цель статьи — прокомментировать 2-киллинговы векторные поля, являющиеся расширением киллинговых векторных полей. Изучаются некоторые алгебраические структуры

этих векторных полей и проводятся различные сравнения с киллинговыми векторными полями, фундаментальными в изучении симметрий. Приведен ряд вычислений, высказано несколько замечаний, касающихся данных векторных полей. Исследовано существование 2-киллинговых векторных полей на конкретных 4-мерных многообразиях, на основе чего построены некоторые примеры, а именно установлены векторные поля, которые являются 2-киллинговыми, но не киллинговыми. Кроме того, доказано, что множество 2-киллинговых векторных полей не удовлетворяет некоторым основным свойствам, таким как образование векторного пространства или алгебры Ли, а также коллинеация кривизны.

Ключевые слова: 2-векторное поле Киллинга, векторное поле Киллинга, алгебра Ли, 4-мерные многообразия

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