connection to be vanishing. There are discribed parallel displacements of Bortolotti′s hyperplane in the connections of the both types, which are degenerate.

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THREE THEOREMS ON PROJECTIVE SUBMERSIONS

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Projective mappings have been extensively studied in the literature. The theory of projective submersions is less investigated (see for example [1] - [4]). This paper is devoted to study of the global theory and the local theory of projective submersions. In particular, we generalize two results from [2] and [4] to a noncompact Riemannian manifold.

1. Introduction

A *submersion* of an m-dimensional pseudo-Riemannian manifold (M, g) onto an n-dimensional (m > n) pseudo-Riemannian manifold (N, g′) is a C^∞ - surjective map π: (M, g) $\to$ (N, g′) such that at each point $x \in M$ the induced tangential map $\pi_{*x} : T_xM \to T_{\pi(x)}N$ is of maximal rank. The inverse image $\pi^{-1}(y)$ of a point $y \in N$ is said to be a *fibre* of π. For a submersion π: (M, g) $\to$ (N, g′), the implicit function theorem tells us that the fibres of π are closed submanifolds of (M, g) and at each point y of (N, g′) $\dim \pi^{-1}(y) = m - n$.

Let g^v denotes the metric of $\pi^{-1}(y)$ induced by g. If $\det (g^v) \neq 0$ then $\pi^{-1}(y)$ is called (see [5]) a *nondegenerate submanifold* of (M, g). Fibres of π which we discuss in this paper are assumed to be nondegenerate. Then a foliation V of (M, g) is given by $V = \{ \pi^{-1}(y) \mid y \in N \}$, which determines the *almost product structure*

$$TM = \ker \pi_* \oplus \ker \pi_*^{\perp}$$

where $\ker \pi_*$ will always be integrable and will be called the *vertical distribution* and $\ker \pi_*^{\perp}$ will be called the *horizontal distribution*.

Consider a curve $\gamma : J \to (M, g)$ in (M, g), where J is being an interval, and denote $\pi (\gamma) = \pi \circ \gamma : J \to (N, g')$ the image of γ by π. Then for each a curve γ which is tangent to the distribution $\ker \pi_*$ we have $\pi (\gamma)$ to be a point in (N, g′).

The curve γ is called *geodesic* if its tangent vector field $\dot{\gamma}$ is parallel, i. e. $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ where ∇ denotes the Levi-Civita connection. That it follows (see [6]) that a geodesic is either regular at every point, or its image degenerates to a point. A *pregeodesic* is a curve γ , which becomes a geodesic after a change of parameter (see [5]). A parameter making the curve γ geodesic is called *affine*.

We assume that an arbitrary pregeodesic in (M, g) is mapped by π into a pregeodesic in (M, g). Such a submersion π is said to be *projective* (cf. [7]). Note also that a pregeodesic γ which is tangent to the distribution ker π_* map to a point of the manifold (N, g′). This does not contradict the definition of a projective submersion.

Following [8], we call a manifold (M, g) a *twisted - product* $M_1 \times_f M_2$ if (M, g) = ($M_1 \times M_2$, $g_1 + f g_2$) for pseudo-Riemannian manifolds (M_1, g_1), (M_2, g_2) and a positive function f: $M_1 \times M_2 \to (0, \infty)$. If π_1 and π_2 are the canonical projections onto the factor (M_1, g_1) and (M_2, g_2), then for any point $y \in M_2$ the fibre $\pi_2^{-1}(y)$ is a totally geodesic submanifold of (M, g), while for any point $z \in M_1$ the fibre $\pi_1^{-1}(z)$ is totally umbilical submanifold of (M, g). The following theorem holds.

THEOREM 1. *Let $\pi : (M, g) \to (N, g')$ be a projective submersion of a simply connected and complete pseudo-Riemannian manifold (M, g) onto a pseudo-Riemannian manifold (N, g′) of smaller dimension. If each fibre of π is a nondegenerate submanifold of (M, g) then (M, g) is isometric to a twisted product $M_1 \times_f M_2$ such that the integrable manifolds of ker π_* and ker $\pi_*^{\perp}$ correspond to the canonical foliations of the product $M_1 \times M_2$.*

We call a manifold (M, g) a *locally twisted product* (see [9]) if for any point $x \in M$ there exists a neighbourhood $U = U_1 \times U_2$ with a local coordinate system $x^1, ..., x^n, x^{n+1}, ..., x^m$ such that the metric form of (M, g) can be written as

$$ds^2 = g_{ab}(x^1, ..., x^n)\, dx^a \otimes dx^b + f(x^1, ..., x^m)\, g_{\alpha\beta}(x^{n+1}, ..., x^m)\, dx^\alpha \otimes dx^\beta$$

for a, b = 1, ..., n; $\alpha, \beta = n + 1, ..., m$ and a certain positive function f. In this case we have proved (see [2]) that the canonical projection $\pi_1 : U_1 \times U_2 \to U_2$ is a projective submersion.

Let $\pi : (M, g) \to (N, g')$ be a projective submersion from a complete Riemannian manifold (M, g) of a nonnegative sectional curvature K onto a Riemannian manifold (N, g′). In this case we have proved (see [2] and [4]) that (M, g) is locally a Riemannian product, that is, the orthogonal distribution ker $\pi_*^{\perp}$ is also integrable with totally geodesic leaves and (M, g) is locally a Riemannian product of the leaves of ker π_* and ker $\pi_*^{\perp}$.

Now we assume that (M, g) is not a complete manifold and present the following local

THEOREM 2. *Let $\pi : (M, g) \to (N, g')$ be a projective submersion of a Riemannian manifold (M, g) of nonnegative sectional curvature onto a Riemannian manifold (N, g′) of smaller dimension. Then the leaves of ker π_* are totally geodesic, the orthogonal distribution ker $\pi_*^{\perp}$ is also integrable with totally geodesic leaves and (M, g) is locally a Riemannian product of the leaves of ker π_* and ker $\pi_*^{\perp}$.*

A Riemannian manifold is called a manifold of *quasi-positive sectional curvature* if its sectional curvatures are everywhere nonnegative (resp. positive semi-definite, etc.) and positive (resp. positive definite, etc.) for all 2-planes at one point. *Quasi-negativity* is dually defined (see for example [10]).

In [3] we have proved that a compact oriented Riemannian manifold of quasi-negative sectional curvature admits no projective submersions onto Riemannian manifolds of smaller dimension. Finally we shall prove

THEOREM 3. *A Riemannian manifold (M, g) of quasi-positive sectional curvature admits no projective submersions onto Riemannian manifolds of smaller dimension.*

Theorem 3 is a purelly local result.

2. Notes on the Proof of Theorem 1

Let $\pi : (M, g) \to (N, g')$ be a submersion from a pseudo-Riemannian manifold (M, g) onto a pseudo-Riemannian manifold (N, g′) of smaller dimension. Now we assume that all fibres of π are nondegenerate submanifolds of (M, g). In this case the foliation $V = \{ \pi^{-1}(y) \mid y \in N \}$ corresponds a pair of mutually complementary orthogonal distributions $\ker \pi_*$ and $\ker \pi_*^{\perp}$.

If π is a projective submersion, then there exists a smooth 1-form θ on (M, g) such that for any two vector fields $X, Y \in C^\infty TM$ we have (see [11])

$$\nabla'_X \pi_* Y - \pi_* \nabla_X Y = \Theta (X) \pi_* Y + \Theta (Y) \pi_* X \qquad (2.1)$$

where ∇' is the Levi-Civita connection of (N, g′) and $\pi_* Y$ is differentiated as a vector field along π. Then, by (2.1), if $X, Y \in C^\infty(\ker \pi_*)$, we get $\pi_* \nabla_X Y = 0$. In this case we have Q = 0 for the second fundamental form Q of V. Hence V is a totally geodesic foliation of (M, g).

On the other hand, if we set $X, Y \in C^\infty(\ker \pi_*^{\perp})$ and $Z \in C^\infty(\ker \pi_*)$ in (2.1), we get

$$\pi_* \nabla_Z X = - \theta(Z) \pi_* X. \qquad (2.2)$$

Then $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) = \theta(Z) g(X, Y)$ and by symmetry g([X, Y], Z) = 0 so that $\ker \pi_*$ is integrable. Moreover g(Q(X, Y), Z) = θ(Z) g(X, Y). Hence $\ker \pi_*^{\perp}$ defines a totally umbilical foliation H.

Let (M, g) be a simply connected pseudo-Riemannian manifold with two complementary orthogonal foliations V and H such that the leaves of V are totally geodesic and geodesically complete and the leaves of H are totally umbilic. In this case R. Ponge and H. Reckziegel proved (see [8]) that (M, g) is isometric to a twisted product $M_1 \times_f M_2$ such that H and V correspond to the cannonical foliations of the product $M_1 \times M_2$.

If we suppose the geodesic completeness of (M, g), then the leaves of V are geodesically complete automatically. Hence the result of R. Ponge and H. Reckziegel implies Theorem 1 immediately.

REMARK . Let (M, g) be a Riemannian manifold and π : (M, g) $\to$ (N, g′) be a projective submersion. In this case the following condition is unnecessary in the text of Theorem 1: fibres of π are nondegenerate.

3. Proofs of Theorem 2 and Theorem 3

Let π: (M, g) $\to$ (N, g′) be a projective submersion of an m-dimensional Riemannian manifold (M, g) onto an n-dimensional Riemannian manifold (N, g′) and m > n. Then the foliation V defined by ker π_* is totally geodesic, the orthogonal distribution ker $\pi_*^{\perp}$ is integrable and defines a totally umbilical foliation H.

The mean curvature vector ξ^h of H defined by g(ξ, Z) = θ(Z) for $Z \in C^{\infty}(\ker \pi_*^{\perp})$ is tangent to V. Then we may consider its divergence $\mathrm{div}_V\, \xi^h$ on V. By Corollary 2.9 of [12] we have

$$4\sum_{a=1}^{n}\sum_{i=n+1}^{m} K(X_a, X_i) = -\frac{1}{2}\,|\nabla P|^2 + 2\,\mathrm{div}_V\,\xi^h , \qquad (3.1)$$

where { $X_1, \dots , X_n$ } and { $X_{n+1}, \dots , X_m$ } are local orthonormal frams of ker $\pi_*^{\perp}$ and ker π_* respectively, K(X_a, X_i) is the sectional curvature of the plane p = span { X_a, X_i } and P is the fundamental tensor of the almost product structure TM = ker $\pi_* \oplus$ ker $\pi_*^{\perp}$.

It is know that $\pi^{-1}(y)$ is a closed submanifold of (M, g). Applying to (3.1) the Green's Theorem we get

$$\int_{\pi^{-1}(y)} \{\sum_{a=1}^{n}\sum_{i=n+1}^{m} K(X_a, X_i) + \frac{1}{8}\,|\nabla P|^2\}\eta = 0, \qquad (3.2)$$

where η is the volume form on $\pi^{-1}(y)$ determined by the metric tensor field g^v .

REMARK . In the case when $\pi^{-1}(y)$ is disconnected, we can apply the Green's Theorem to each its connected component.

We assume that $K(X_a, X_i) \geq 0$ for $1 \leq a \leq n$ and $n + 1 \leq i \leq m$. Then using (3.2) we obtain $K(X_a, X_i) = 0$ and $\nabla P = 0$ at each poin of $\pi^{-1}(y)$.

The foliation V = { π^{-1} (y) | y $\in$ N } is a decomposition of (M, g) into a union of disjoint closed submanifolds $M = \bigcup_{y \in N} \pi^{-1}(y)$, so that $\nabla P = 0$ at each point of (M, g).

Hence (see for example [9]) the orthogonal distributions ker π_* and ker $\pi_*^{\perp}$ are integrable with totally geodesic leaves. Consequently, (M, g) is locally a Riemannian product of leaves of ker π_* and ker $\pi_*^{\perp}$.

Assume now that $K \geq 0$ and $K > 0$ for some point x of (M, g) where $x \in \pi^{-1}\pi(x)$. In this case from (3.2) is conluded, that the submersion π: (M, g) $\to$ (N, g′) can not be projective.

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С.Е. С т е п а н о в, И.И. Ц ы г а н о к

ТРИ ТЕОРЕМЫ О ПРОЕКТИВНЫХ СУБМЕРСИЯХ

Проективные отображения широко изучены в литературе. Теория проективных субмерсий менее исследована. Настоящая работа посвящена изучению глобальной и локальной теорий проективных субмерсий. В частности, обобщаются 2 результата в случае некомпактного риманова многообразия.

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СУЖЕНИЯ ПРОСТРАНСТВ ПРОЕКТИВНОЙ СВЯЗНОСТИ, ИНДУЦИРУЕМЫХ НА ОСНАЩЕННОЙ ГИПЕРПОЛОСЕ

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