MSC 2010: 53A45, 53C20

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doi: 10.5922/0321-4796-2024-55-2-2

Two kernel vanishing theorems and an estimation theorem for the smallest eigenvalue of the Hodge — de Rham Laplacian

In this paper, we formulate two theorems on the disappearance of the kernel of the Hodge — de Rham Laplacian and refine the estimate for its smallest eigenvalue on closed Riemannian manifolds.

***Keywords*:** Riemannian manifold, exterior differential form, Hodge — de Rham Laplacian, kernel vanishing theorem, smallest eigenvalue

1. Definitions and notations

In this paper, we will consider the *Hodge — de Rham Laplacian* $\Delta_H: C^\infty(\Lambda^q M) \to C^\infty(\Lambda^q M)$, where $\Lambda^q M$ is the vector bundle of exterior differential q-forms $(1 \le q \le n-1)$ over an n-dimensional Riemannian manifold (M, g).

Next, let (M, g) be covered by a system of coordinate neighborhoods $\{U, x^1, \dots x^n\}$, where U denotes a neighborhood and $x^1, \dots x^n$ denote local coordinates in U. Then we can define the natural frame $\{X_1 = \partial/\partial x^1, \dots, X_n = \partial/\partial x^n\}$ in an arbitrary coordinate neighborhood $\{U, x^1, \dots x^n\}$. In this case, $g_{ij} = g(X_i, X_j)$ are local components of the metric tensor g with the indices $i, j, k, l, \dots \in \{1, 2, \dots, n\}$. Next, we denote by R_{ik} and R_{ikjl} the lo-

Submitted on February 16, 2024

cal components the Ricci tensor Ric and the curvature tensor R, respectively. Then the Hodge — de Rham Laplacian $\Delta_H\colon C^\infty(\Lambda^q M) \to C^\infty(\Lambda^q M)$ with respect to local coordinates $x^1, \dots x^n$ has the form

$$\Delta_H \omega_{i_1 \dots i_q} = \bar{\Delta}\omega_{i_1 \dots i_q} + \mathfrak{R}_p(\omega)_{i_1 \dots i_q},$$

where $\bar{\Delta} = -\operatorname{trace}_g \nabla^2$ and (see, e. g., [1])

$$\mathfrak{R}_q(\omega)_{i_1 \dots i_p} = \sum_{a=1}^{q} g^{jk} R_{i_a j} \omega_{i_1 \dots i_{a-1} k i_{a+1} \dots i_p} -$$

$$- \sum_{\substack{a,b=1 \\ a \neq b}}^{q} g^{jk} g^{lm} R_{i_a i_b j l}\, \omega_{i_1 \dots i_{a-1} k i_{a+1} \dots i_{b-1} m i_{b-1} \dots i_p}$$

for $\omega \in \Lambda^q M$. In particular, $\mathfrak{R}_1 = Ric$. In this case, direct calculations yield the following formula:

$$\frac{1}{2} \Delta \|\omega\|^2 = -g(\Delta_H \omega, \omega) + g\big(\mathfrak{R}_q(\omega), \omega\big) + \|\nabla\omega\|^2,$$

where $\Delta = \operatorname{trace}_g \nabla^2$ and (see, e. g., [2])

$$g\big(\mathfrak{R}_q(\omega), \omega\big) = q\left(R_{ij}\omega^{i i_2 \dots i_q}\omega^{j}_{i_2 \dots i_q} - \tfrac{q-1}{2} R_{ikjl}\omega^{iki_3 \dots i_q}\omega^{jl}_{i_3 \dots i_q}\right).$$

In particular, we have

$$\Delta \|\omega\|^2 \geq 2\, g\big(\mathfrak{R}_q(\omega), \omega\big) \tag{1}$$

for an arbitrary $\omega \in \Lambda^q M \cap \ker\Delta_H$. We recall that on a closed Riemannian manifold, by the Hodge's theorem the dimension of the kernel of $\Delta_H\colon C^\infty(\Lambda^q M) \to C^\infty(\Lambda^q M)$ equals the q^{th} Betti number $\mathrm{b}_q(M)$, and so the Laplacians determine the Euler characteristic $\chi(M)$.

We recall that the curvature tensor induces a self-adjoint operator $\hat{R}\colon \Lambda^2 M \to \Lambda^2 M$, defined by the equations, see [3], $\hat{R}(\omega)_{ij} = R_{ijkl}\omega^{kl}$ for an arbitrary $\omega \in \Lambda^2 M$. The map $\hat{R}\colon \Lambda^2 M \to \Lambda^2 M$, called the *curvature operator of the first kind*, see [3; 4], induces a bilinear form $\hat{R}\colon \Lambda^2 M \times \Lambda^2 M \to \mathbb{R}$ by restriction to $\Lambda^2 M$. We say that $\hat{R} > 0$ if the eigenvalues of $\hat{R}$ as a bilinear form on $\Lambda^2 M$ are positive (the bilinear form is positive definite). Moreover, if $\hat{R}$ is positive definite at each point $x \in M$, then $\mathfrak{R}$ is also positive definite at each point $x \in M$. In addition, if $\hat{R}$ is positive semi-definite at each point $x \in M$, then so is $\mathfrak{R}$.

2. Two kernel vanishing theorems for the Hodge — de Rham Laplacian

Based on (1) and the above statements, we can formulate the classical vanishing theorem on the disappearance of the kernel Δ_H (see [5, p. 351; 6, p. 334; 7, p. 336–337]).

Theorem 1. *Let Δ_H be the Hodge — de Rham Laplacian defined on C^∞-sections of the fibre bundle of exterior differential q-forms $(1 \leq q \leq n-1)$ over a closed n-dimensional Riemannian manifold (M, g). If the curvature operator of the first kind $\hat{R}: \Lambda^2 M \to \Lambda^2 M$ of (M, g) is positive semi-definite, then $\nabla \varphi = 0$ for an arbitrary $\varphi \in \ker \Delta_H$ and $\dim_{\mathbb{R}} ker\, \Delta_H = \mathfrak{b}_q(M) \leq \binom{n}{q}$. In particular, if $\hat{R}$ is positive definite at each point $x \in M$, then*

$$dim_{\mathbb{R}} \ker \Delta_H = \mathfrak{b}_q(M) = 0.$$

Remark. We recall that Böhm and Wilking showed by Ricci-flow techniques that positive curvature operator $\hat{R}$implies that a closed manifold (M, g) is diffeomorphic (not isometric) to a spherical space form (see [8]).

For the case of a complete and non-compact Riemannian manifold, we deduce the following statement from our inequality (1), Theorem 3 and Theorem 7 from [9].

Theorem 2. *Let Δ_H be the Hodge — de Rham Laplacian defined on C^∞-sections of the fibre bundle of exterior differential q-forms over a complete and non-compact n-dimensional Riemannian manifold (M, g) for $(1 \leq q \leq n-1)$. If the curvature operator of the first kind $\hat{R}: \Lambda^2 M \to \Lambda^2 M$ of (M, g) is positive semi-definite, then $L^k(Ker\Delta_H)$ is trivial for any number $k > 1$.*

Remark. Our statement above generalizes the following now-classic result from [10] and [11]: If $\mathfrak{R}_q$ is positive semi-define at every point of a complete Riemannian manifold (M, g), then L^2-harmonic q-form is parallel. In particular, if either exists a point $x \in M$ such that $\mathfrak{R}_q$ is strictly positive at x or the volume of (M, g) is infinite, then every L^2-harmonic q-form is identically zero.

3. An estimation theorem for the smallest eigenvalueof the Hodge — de Rham Laplacian

Having discussed the kernel of the Hodge Laplacian Δ_H, we now turn our attention to its first positive eigenvalue on closed Riemannian manifold, which we will denote by $\lambda_1^{[q]}$. Note here that the superscript $[q]$ refers to the degree of the involved eigenform. We also recall that the spectrum $Spec^{(q)}\Delta_H$ of the Hodge Laplacian consists only of non-negative eigenvalues with finite multiplicity. We also denote its positive eigenvalues counted with multiplicity by

$$0 = \lambda_0^{[q]} < \lambda_1^{[q]} \leq \lambda_2^{[q]} \leq \cdots \leq \lambda_k^{[q]} \leq \lambda_{k+1}^{[q]} \leq \cdots,$$

where the multiplicity of the eigenvalue 0 is equal to the q-th Betti number $\mathfrak{b}_q(M)$ of (M, g), by the Hodge — de Rham theory (see, for example, [5; 7, p. 339]). The case $q = 0$ corresponds to the *Laplace — Beltrami* $\Delta = \delta\nabla$ operator acting on C^∞-functions. At the same time, we known from [12, p. 78] that if all eigenvalues of $\hat{R}$ lie in $[\hat{r}_{min}, \hat{r}_{max}]$, then the sectional curvature sec satisfies $1/2\hat{r}_{min} \leq sec \leq 1/2\hat{r}_{max}$. Therefore, if the inequality $\hat{R} \geq C > 0$ holds and then, from the above, we have $sec \geq 1/2\ C$. In this case, $Ric \geq 1/2\,(n-1)C$, and, as Lichnerowicz has already proved, $\lambda_q^{[0]} \geq 1/2\, n\, C$ (see, for example, [7, p. 82]). A similar result can be formulated about the eigenvalues $\bar{\lambda}_a^{[q]}$ and $\lambda_q^{[q]}$ of the Laplacians $\bar{\Delta}$ and Δ_H, respectively. But let us first recall the following.

The variational characterization of the eigenvalues $\bar{\lambda}_a^{[q]}$ and $\lambda_q^{[q]}$ of the Laplacians $\bar{\Delta}$ and Δ_H will be as follows (see [13, p. 393]):

$$\lambda_q^{[q]} \geq \bar{\lambda}_a^{[q]} + \mathfrak{R}_{min} \tag{2}$$

for all $a \geq 1$. Here, since (M, g) is closed, we have defined the number (see [13, p. 379])

$$\mathfrak{R}_{min} = \inf\{\mathfrak{R}_{min}(x) : x \in M\}$$

for $\mathfrak{R}_{min}(x) = \inf\{g(\mathfrak{R}\varphi, \varphi)_x : \varphi_x \in E_x,\ g(\varphi, \varphi)_x = 1\}$. In addition, we recall that the rough Laplacian $\bar{\Delta}$ acting on $C^\infty(E)$ is an order 2 elliptic operator and that its spectrum on a closed (M, g) is an unbounded sequence of real numbers $Spec^{(0)}\bar{\Delta}= \{\bar{\lambda}_a\}_{a\in\mathbb{N}}$ which can be increasingly ordered (see [14])

$$0 = \bar{\lambda}_0^{[q]} \leq \bar{\lambda}_1^{[q]} \leq \cdots \leq \bar{\lambda}_k^{[q]} \leq \bar{\lambda}_{k+1}^{[q]} \leq \cdots$$

with the following convention: $\bar{\lambda}_0^{[q]}$ is the zero eigenvalue with multiplicity $\dim(Ker\nabla)$. In case where there is no parallel section, i. e., $\dim(Ker\nabla) = 0$, the spectrum starts with the positive eigenvalue $\bar{\lambda}_1$.

Theorem 3. *Let (M, g) be an n-dimensional closed Riemannian manifold. Let $\bar{\Delta}$ and Δ_H be the rough and Hodge — de Rham Laplacians actingdefined on C^∞-sections of the fibre bundle $\Lambda^q M$ of differential q-forms,* $1 \leq q \leq n-1$. *If the curvature operator of the second kind $\hat{R}: \Lambda^2 M \to \Lambda^2 M$ satisfies the inequality $\hat{R} \geq C > 0$ and (M, g) is not isometric to the Euclidean n-sphere $\mathbb{S}^n$ with its standard metric, then $\bar{\lambda}_a^{[q]} > 1/2\ n\, C$ and $\lambda_q^{[q]} > 1/2\ n\, C +$ $+\, q(n-q)\, C$ for any eigenvalues $\bar{\lambda}_a^{[q]}$ and $\lambda_q^{[q]}$ of $Spec^{(q)}\bar{\Delta}$ and $Spec^{(q)}\Delta_H$, respectively.*

Proof. Let (M, g) be an n-dimensional closed Riemannian manifold. We recall that if there exists a positive constant C on (M, g) such that the inequality $g(\hat{R}\omega, \omega) \geq C\|\omega\|^2$ holds for any 2-form $\omega \in \Lambda^2 M$, then the inequality $g(\mathfrak{R}_q(\omega), \omega) \geq$

$\geq q(n-q)\, C\, \|\omega\|^2$ holds for any $\omega \in \Lambda^q M$ and $1 \leq q \leq n-1$ (see [15]). In addition, equality holds for a Riemannian manifold isometric to the Euclidean n-sphere $\mathbb{S}^n$ with its standard metric. In other words, C serves here as a lower bound for the eigenvalues of the curvature operator of the first kind $\hat{R}$ of (M, g) and, in turn, $\mathfrak{R}_{min} = q(n-q)\, C$ serves here as a lower bound for the eigenvalues of the Weitzenböck curvature operator $\mathfrak{R}_q$ of (M, g), respectively. In this case the variational characterization of the eigenvalues (2) is as follows

$$\lambda_a^{[q]} \geq \bar{\lambda}_a^{[q]} + q(n-q)C \tag{3}$$

for any $\lambda_a^{[q]} \in Spec^{(q)}\Delta_H$ and $\bar{\lambda}_a^{[q]} \in Spec^{(q)}\bar{\Delta}$. In contrast to the Hodge Laplacian Δ_H, the kernel of the rough Laplacian $\bar{\Delta}$ acting on q-forms consists of parallel q-forms, whose dimension is not a topological invariant. Therefore, if $\bar{\lambda}_a^{[q]} = 0$, then the associated eigenspace consists of parallel q-forms. At the same time, it is well-known that there are no parallel q-forms $(1 \leq q \leq n-1)$ on a closed Riemannian manifold with a positive curvature operator of the first kind $\hat{R}$ (see [5, p. 351]). Therefore, in our case, $\bar{\lambda}_a^{[q]} \neq 0$, i. e., for the metric g with $\hat{R} \geq C > 0$, all eigenvalues of the rough and Hodge Laplacians acting on q-forms, $1 \leq q \leq n-1$, are non-zero.

At the same time, for the metric g with $\hat{R} \geq C > 0$ and every q, $1 \leq q \leq 1/2\, n$, inequality (3) can be rewritten as the first Gallot — Meyer inequality (see [16] and inequality (3.4) from [17])

$$\lambda_a^{[q]} \geq qC + q(n-q)C.$$

In turn, for the metric g with $\hat{R} \geq C > 0$ and every q, $1/2\, n \leq$ $\leq q \leq n-1$, inequality (3) can be rewritten as the second Gallot — Meyer inequality (see, for example, inequality (3.3) from [16])

$$\lambda_a^{[q]} \geq (n-q)C + q(n-q)C,$$

because $1 \leq (n-q) \leq 1/2\, n$. Moreover, two lower bounds of $\lambda_a^{[q]}$ are optimal because they can be achieved for a Riemannian manifold (M, g) isometric to the Euclidean n-sphere $\mathbb{S}^n$ with its standard metric; in other words, for this variety the equalities are valid in both cases (see also [7, p. 342]). Therefore, if a Riemannian manifold (M, g) isometric to the Euclidean n-sphere $\mathbb{S}^n$, then the first Gallot — Meyer inequality can be rewritten as the equality $\lambda_a^{[q]} = q\ C + q(n - q)\ C$ for every q, $1 \leq q \leq 1/2\, n$. In this case, from (3) we deduce $\bar{\lambda}_a^{[q]} \leq 1/2\ n\ C$ for all $a \geq 1$. A similar conclusion can be drawn for the case when $1/2\, n \leq q \leq n - 1$. Therefore, $\bar{\lambda}_a^{[q]} > 1/2\ n\ C$ for an n-dimensional closed Riemannian manifold with $\hat{R} \geq C > 0$ and not isometric to the Euclidean n-sphere $\mathbb{S}^n$. Then from (3) we deduce that $\lambda_q^{[q]} > 1/2\ n\ C + q(n - q)\ C$. Then our theorem holds.

Remark. If $\bar{\lambda}_a^{[q]} > 1/2\ n\ C$, then both strictly Gallot — Meyer inequalities will automatically follow from (3) for an n-dimensional closed Riemannian manifold with $\hat{R} \geq C > 0$ and not isometric to the Euclidean n-sphere $\mathbb{S}^n$.

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For citation: Arkhipova, N. A., Stepanov, S. E. Two kernel vanishing theorems and an estimation theorem for the smallest eigenvalue of the Hodge — de Rham Laplacian. DGMF, 55 (2), 37—46 (2024). https://doi.org/10.5922/0321-4796-2024-55-2-2.

УДК 53A45; 53C20

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doi: 10.5922/0321-4796-2024-55-2-2

Две теоремы исчезновения и теорема оценки наименьшего собственного значения лапласиана Ходжа — де Рама

Поступила в редакцию 16.02.2024 г.

В данной работе рассматривается лапласиан Ходжа — де Рама. Формулируются две теоремы об исчезновении ядра лапласиана Ходжа — де Рама. Уточняется оценка наименьшего собственного значения лапласиана на замкнутых римановых многообразиях.

Ключевые слова: риманово многообразие, внешняя дифференциальная форма, лапласиан Ходжа — де Рама, теорема исчезновения ядра, наименьшее собственное значение

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Для цитирования: *Архипова Н.А., Степанов С.Е.* Две теоремы исчезновения и теорема оценки наименьшего собственного значения лапласиана Ходжа — де Рама // ДГМФ. 2024. №55 (2). С. 37—46. https://doi.org/10.5922/0321-4796-2024-55-2-2.