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A SURPRISING TALE OF LONG-PERIODIC SPIN OSCILLATIONS IN THE SYNTHETIC ANTIFERROMAGNETS: SOME EXACT SOLUTIONS

This article is devoted to construction of a mathematical theory capable of explaining those experimentally observable periodic magnetic oscillations in the synthetic antiferromagnet Pt/Co/Ir/Co/Pt that take place after a switch in the direction of an external magnetic field. In particular, we demonstrate that in order to understand the aforementioned phenomenon it is essential to first properly model the collisions between the magnetic domains of different spin orientations (P^- and AP^-). The resulting model comprised of a system of nonlinear differential equations is closely examined, after which we propose a simple analytical method of construction of its exact solutions. This method is shown to generate an infinite family of solutions associated with the degenerate hypergeometric functions, parameterized by a natural number N. One of those solutions with $N = 2$ produces the magnetization function which perfectly fits the experimental data.

Данная работа посвящена построению математической теории, корректно объясняющей экспериментально наблюдаемые периодические осцилляции намагниченности в синтетическом антиферромагнетике Pt/Co/Ir/Co/Pt при изменении направления внешнего магнитного поля. Показано, что существенную роль в модели играют столкновения друг с другом магнитных доменов различной спиновой ориентированности (P^- и AP^-). Обсуждены особенности полученной системы нелинейных дифференциальных уравнений и предложен простой аналитический метод построения бесконечного множества решений этой системы, выраженных через специальным образом выбранные вырожденные гипергеометрические функции, параметризованные целым числом N. Показано, что решение с $N = 2$ в точности совпадает с экспериментальной кривой намагниченности.

Keywords: synthetic antiferromagnets, domain walls, Schrödinger equation, degenerate hypergeometric equation.

Ключевые слова: синтетические антиферромагнетики, доменные стенки, уравнение Шрёдингера, вырожденное гипергеометрическое уравнение.

Introduction

Ever since the pioneering work on the potential coupling between the adjacent layers of different magnetic materials, published in 1986 (see [1—4]), the subject of multilayered ferro-, ferri, and anti-ferromamagnetics has never left the limelight. The realization of the fact that a subtle change in a thicknesses of a non-magnetic material — the «spacer» between two (or more) layers of

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ferromagnets — can totally change the character of the interaction between those layers and switch it between non-existent (a very thick spacer) to ferromagnetic to antiferromagnetic came as a storm. Out of this storm a number of very interesting devices has emerged, including the spin valves and the synthetic antiferromagnets (SAF). The range of current and prospective applications of such devices is staggering; it ranges from the magnetic random access memory [5; 6] to the sensors for various biomedical applications [7; 8].

Despite the fact that a general mechanism at work in the SAF — the Ruderman — Kittel — Kasuya — Yosida (RKKY) coupling between the magnetic domains in different layers has been known since the end of 1950-s (see [9; 11]), the particulars of the behavior of magnetization in SAF are still capable of puzzling the scientists. One such enigma has been described in [12]: a very unusual non-monotonous relaxation pattern in a Pt/Co/Ir/Co/Pt multilayered SAF after the switching of the direction of the external magnetic field. This was rather unexpected as the equations normally used to describe the dynamics of the magnetic domains in SAF were all linear and did not predict the dynamics observed in [12].

A year later the answer has been found [13]: the culprit was shown to be the collisions between the different magnetic domain in the magnetic layers. This article serves as a mathematical supplement to [13] and is designed to provide a detailed mathematical exploration of the subject whereas [13] is dedicated mostly to the experimental and physical side of the research.

The statement of the problem

Our goal for this article would be to study the behavior of the coupled magnetic domains in the SAF with a perpendicular anisotropy after the switching of the external magnetic field. For certainty we will assume that the thicker of the two magnetic layers — the "anchor" — is the lowest of the two. A total of four types of magnetic domains are possible: two parallel states (P^+ with spins in both magnetic layers looking up and P^- where the spins point downwards) and two antiparallel (AP^+ and AP^- — the sign determined by the spin orientation in the anchor) — see Fig. 1.

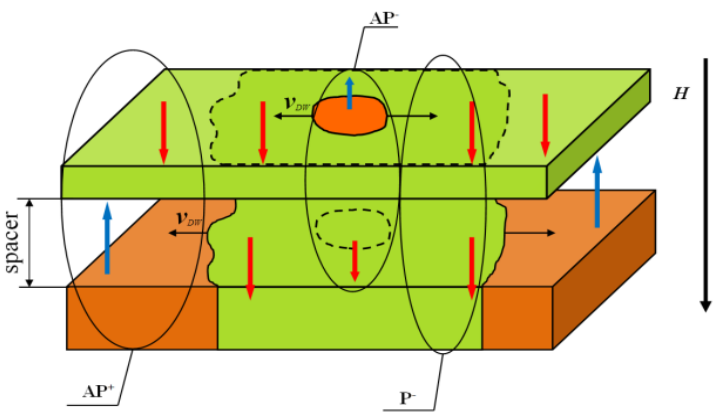

Fig. 1. The scheme of the SAF and the main types of the magnetic domains therein

Initially the SAF in its entirety consists of just one magnetic domain – the AP^+ type (for that end we switch on the external magnetic field and wait until the magnetization of SAF stabilizes). However, if we should turn on the external magnetic field again, but this time of opposite direction, the new types of domains will start emerging: the nuclei of the P^- and the AP^- phases. But, of course our sample being SAF, only one type — the AP^- will survive in the end.

With that being said, let us look at the simplest model of the dynamics of those three types of domains.

Let z be a concentration density of AP^+ nuclei;

x — a density of P^- nuclei;

y — a density of the AP^- nuclei;

$\alpha(H,T)$ — an efficiency of the P^- nuclei generation from the AP^+ phase;

$\beta(H,T)$ — an efficency of AP^- nuclei generation from the P^- phase;

$\gamma(H,T)$ — an efficiency of the AP^- nuclei generation from the AP^+ phase.

Naturally, since the P^- phase is only transitory, and the total concentration is limited by the size of the sample, we conclude that the rate of growth of AP^- must be proportional to both z (the more AP^+ nuclei means more chance for an AP^- to emerge) and the x (every P^- nuclei has a chance to morph into AP^- nuclei). For the same reason the growth of x shall also be proportional to z but be stifled by the big x (the more of them we have the more will convert to AP^- phase). Finally, the finite size of the sample determines that the sum of all three types of nuclei must be a constant — in our case 1 (since we are working with the concentrations).

This produces the following simple linear system:

$$\begin{cases} \dfrac{dx}{dt} = \alpha z - \beta x \\ \dfrac{dy}{dt} = \gamma z + \beta x \\ x + y + z = 1 \end{cases} \tag{1}$$

with the initial conditions corresponding to the starting P^+ state at $t=0$ are $z=1$, $x=0$, $y=0$. Once we solve this system, calculating the total magnetization of SAF will be an easy task:

$$M(t) = M_{S1}(-x - 0.3y + z),$$

where M_{s1} would be the saturating magnetization of the thick layer.

So, how do we solve (1)? First of all, if we sum up the first two equation in (1) we will end up with

$$\dot{z} = -(\alpha + \gamma)z,$$

whose general solution is

$$z = z_0 e^{-(\alpha+\gamma)t}.$$

Plugging this into the first equation in (1) produces the following equation on x:

$$\dot{x} = \alpha z_0 e^{-(\alpha+\gamma)t} - \beta x,$$

which is an inhomogeneous linear differential equation. Its general solution is:

$$x = \left(x_0 - \frac{\alpha z_0}{\beta - \alpha - \gamma} \right) e^{-\beta t} + \frac{\alpha z_0}{\beta - \alpha - \gamma} e^{-(\alpha+\gamma)t}.$$

This we can of course use in the equation on y. Solving it and using the condition $y_0 = 1 - x_0 - z_0$ yields the following solution

$$y = 1 - \frac{z_0}{\alpha + \gamma} \left(\gamma + \frac{\alpha\beta}{\beta - \alpha - \gamma} \right) e^{-(\alpha+\gamma)t} - \left(x_0 - \frac{\alpha z_0}{\beta - \alpha - \gamma} \right) e^{-\beta t}.$$

The resulting magnetization M together with the experimental results are shown on Figure 2.

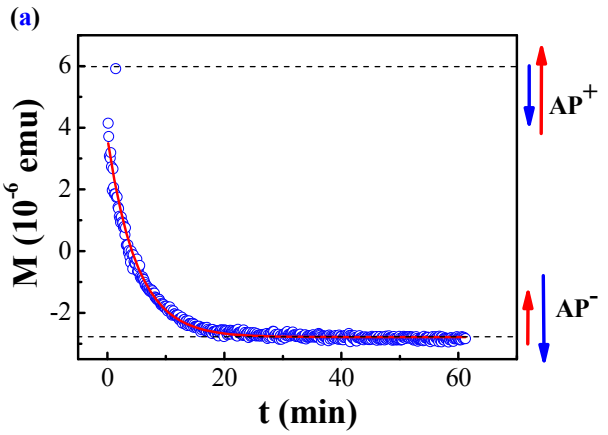

Fig. 2. Magnetic relaxation in Pt(3 nm)/Co(1.05 nm)/Ir(1.5 nm)/Co(0.7 nm)/Pt(3 nm) in the magnetic field –1350 Oe at T = 100 K. Here M(t) = M_{S1} (–x – 0.3y + z), where M_{s1} is the saturating magnetization of the thick layer. The solid line is the exact solution of (1), the blue circles — the results of the observations (for further details see [13])

Everything looks good if not for one little thing: while the behaviour of the solution is motonous (owing to the linearity of the system (1)), at some values of the external magnetic field the magnetization's dynamic is ***no longer monotonous***.

This can only mean one thing: at those values the model must be incorrect!

The model with the domain interaction

Let's take into account the fact that in the process of their growth the P- and AP- domains inevitably bump into each other. In the process the P- → AP- transition takes place: a P- phase absorption by AP- phase.

To account for this absorption, we have to add two additional terms, proportional to the probability δxy of the nuclei approaching each other:

$$\begin{cases} \dfrac{dx}{dt} = \alpha z - \beta x - \delta xy \\ \dfrac{dy}{dt} = \gamma z + \beta x + \delta xy \\ x + y + z = 1 \end{cases}' \tag{2}$$

where the coefficient δ describes the efficiency of the P- phase absorption by AP- phase.

The initial conditions at $t=0$ remain the same: $z=1$, $x=0$, $y=0$.

A close look at our system is in order…

If we sum up first and second equations in (2), we'll get:

$$\dot{z} = -(\alpha + \gamma)z\,,$$

whose general solution is

$$z = z_0 e^{-\rho t} \tag{3}$$

and $\rho = \alpha + \gamma$. According to (2), this yields the system:

$$y = 1 - x - z_0 e^{-\rho t}\,, \tag{4}$$

$$\dot{x} = \alpha z_0 e^{-\rho t} + \left[\delta z_0 e^{-\rho t} - \beta - \delta + \delta x\right] x\,. \tag{5}$$

The equation (5) should look familiar to anyone proficient in the theory of O.D.E.'s: it is the famous *Riccati equation* [14]. One of its interesting properties if that this equation is homogenous with respect to the variables $\dot{x}$ and x (but not t), which implies that (5) can be linearized by the following change of variable:

$$x(t) = -\frac{1}{\delta}\frac{d}{dt}\ln f(t), \tag{6}$$

which converts the Riccati equation into an even more famous *Schrödinger equation*. And if we move one step further and additionally rescale $f(t)$ as:

$$f(t) = \exp\left[-\frac{1}{2}\left(\sigma t + \frac{\delta z_0}{\rho} e^{-\rho t}\right)\right] V(t), \qquad \sigma = \beta + \delta\,,$$

then we'll end up with the Schrödinger equation that looks like this:

$$\frac{\ddot{V}}{V} = \frac{\delta^2}{4} e^{-2\rho t} + \frac{\delta}{2}\left[\rho - \sigma - 2\alpha\right] e^{-\rho t} + \frac{1}{4}\sigma^2. \tag{7}$$

Hence, the entire problem reduces to finding a regular solution of (7) which does not vanish for any given t>0 (so that the r.h.s of (7) remains well-defined), and then using it to find x:

$$x(t) = \frac{\sigma}{2\delta} - \frac{z_0}{2} e^{-\rho t} - \frac{1}{\delta}\frac{\dot{V}(t)}{V(t)}. \tag{8}$$

Prior to that, however, it would be handy to replace the time variable t with a new independent variable $\xi = e^{-\rho t}$. After that, the Schrödinger equation assumes a much simpler form:

$$V'' + \frac{1}{\xi}V' - \left[a^2 + \frac{ab}{\xi} + \frac{c^2}{\xi^2}\right]V = 0, \tag{9}$$

where for the sake of simplicity we have introduced three new coefficients:

$$a = \frac{\delta}{2\rho}, \qquad b = 1 - \frac{\sigma + 2\alpha}{\rho}, \qquad c = \frac{\sigma}{2\rho}.$$

The goal now would be to study this equation and to find out a way to construct its solutions. But we once again remind our reader, that we also have an additional burden on our shoulders: respecting the physical implications of (6), by making sure that the newly discovered solutions do not vanish (for if they do, the density x of P^- will become singular, and this definitely would *not do*!). How shall we approach this daunting task?

Let us start by figuring out the general behavior of the solutions of (9) at the boundaries of the domain $0 < \xi < +\infty$.

1. What happens with the solutions of (9) around $\xi = 0$?

For sufficiently small ξ (9) turns into

$$V'' + \frac{1}{\xi}V' - \frac{c^2}{\xi^2}V = 0.$$

It is easy to see that this equation has two partial solution $V_1 = \xi^{-c}$ and $V_2 = \xi^{+c}$, so its general solution would be just a linear combination of the two: $V(\xi) = c_1\xi^{-c} + c_2\xi^{+c}$. By assumption, V cannot be equal to zero (so $c_1 \neq 0$), hence when $\xi \to 0$

$$V \to \xi^{-c}.$$

2. Similarly, when $\xi \to \infty$, (9) reduces to

$$V'' - a^2V = 0,$$

whose solutions have the asymptotes

$$V \to e^{\pm a\xi}.$$

Armed with this knowledge, we can utilize a new variable w(ξ), defined as:

$$V = \xi^{-c}e^{ka\xi}w(\xi), \qquad k = \pm 1,$$

which, upon substitution into (9) reduces it to the equation for the *degenerate hypergeometric function* [15]:

$$\frac{d^2w(\zeta)}{d\zeta^2} + (1 - 2c - \zeta)\frac{dw(\zeta)}{d\zeta} - a\lambda(k - 2ck - b)w(\zeta) = 0, \tag{10}$$

where

$$\zeta = -2ka\xi. \tag{11}$$

In order to proceed further, it is advisable to represent the solution of degenerate hypergeometric equation as [16]:

$$V(t) = e^{\frac{\sigma t}{2}}\left[c_1 \exp\left(\frac{\delta}{2\rho}e^{-\rho t}\right)w_+(t) + c_2 \exp\left(-\frac{\delta}{2\rho}e^{-\rho t}\right)w_-(t)\right], \tag{12}$$

where, as before, c_1 and c_2 are arbitrary real constants, the functions w_+ and w_- are defined as:

$$\begin{aligned} w_+(t) &= F\left(\frac{\alpha}{\rho}, 1-\frac{\sigma}{\rho}, -\frac{\delta}{\rho}e^{-\rho t}\right), \\ w_-(t) &= F\left(1-\frac{\sigma+\alpha}{\rho}, 1-\frac{\sigma}{\rho}, \frac{\delta}{\rho}e^{-\rho t}\right) \end{aligned} \tag{13}$$

and the function *F* in (13) is given by the following convergent series:

$$F(A,B,\zeta) = 1 + \frac{A}{B}\frac{\zeta}{1!} + \frac{A(A+1)}{B(B+1)}\frac{\zeta^2}{2!} + \ldots \tag{14}$$

So, what possible benefit might we gain from (12)—(14) which we could not from the original Schrödinger equation (9)? Quite a lot, in fact, since now the problem of regularity of $x(t)$ and $y(t)$ reduces to a question of whether the series (13) has any zeroes or not. And a close look at the series $w_+(t)$ reveals it to be *alternating*! Therefore, if we are to remove any possibility of *V*(*t*) turning to zero (henceforth producing a pole in both $x(t)$ and $y(t)$), we have to remove the $w_+(t)$ from the big picture — which we can do by setting $c_1 = 0$. But even that is not the end of the story.

Since we are left with just one series $w_-(t)$, we can play a little bit with its arguments in (14). In particular, we can turn $w_-(t)$ into a *finite* series; all we have to do for that end is introduce a natural number *N* and choose the parameters *A* and *B* for $F(A,B,\zeta)$ to depend on *N* as

$$\begin{aligned} A &= -N \\ B &= 1 - N - \frac{\gamma}{\rho} \end{aligned} \tag{15}$$

Voila! We end up with an infinite of solutions $V_N(t)$, parameterized via the natural number *N*, each one of them being containing within a *non-vanishing* polynomial of order *N* w.r.t. the variable ζ and having the following form

$$V_N(t) = \exp\left[-\frac{\delta}{2\rho}e^{-\rho t} + \frac{\sigma_N\, t}{2}\right] F(-N, 1-N-\frac{\gamma}{\rho}, \frac{\delta}{\rho}e^{-\rho t}), \qquad (16)$$

where we have defined $\sigma_N = \gamma + N\rho$ and $F(\ldots)$ is the series defined by (14).

Since we have gone this far, let us take a look at some first iterations of the function V_N:

$$V_1(t) = \exp\left[-\frac{\delta}{2\rho}e^{-\rho t} + \frac{1}{2}(\gamma+\rho)t\right]\left(1+\frac{\delta}{\gamma}e^{-\rho t}\right),$$

$$V_2(t) = \exp\left[-\frac{\delta}{2\rho}e^{-\rho t} + \frac{1}{2}(\gamma+2\rho)t\right]\cdot\left(1+\frac{2\delta}{\rho+\gamma}e^{-\rho t} + \frac{\delta^2}{\gamma(\rho+\gamma)}e^{-2\rho t}\right),$$

$$V_3(t) = \exp\left[-\frac{\delta}{2\rho}e^{-\rho t} + \frac{1}{2}(\gamma+3\rho)t\right],$$

$$\left(1+\frac{3\delta}{2\rho+\gamma}e^{-\rho t} + \frac{3\delta^2}{(2\rho+\gamma)(\rho+\gamma)}e^{-2\rho t} + \frac{\delta^3}{(2\rho+\gamma)(\rho+\gamma)\gamma}e^{-3\rho t}\right).$$

As we can clearly see from these simplest cases already at the second iteration the solution demonstrates a very clear non-monotonic behavior. In particular, that very iteration, $V_2(t)$, corresponds to the following magnetic nuclei concentrations:

$$\begin{cases} x = \dfrac{m^2 z(1-z)\left[(1+2m)z+1\right]}{m(1+2m)z(mz+2)+m^2+4m+2} \\ y = \dfrac{(1-z)\left[m(2+3m)z+m^2+4m+2\right]}{m(1+2m)z(mz+2)+m^2+4m+2} \\ z = e^{-\rho t} \end{cases}, \qquad (17)$$

where $\dfrac{\delta}{\gamma} = m$.

If we them use (16) to calculate the total magnetization $M(t)$, we will get the following:

$$M = \frac{-3\left(m^2+4m+2\right)+z\left[2\left(3m^2-23m-13\right)+mz\left(20m^2-49m-26-20m(1+2m)z\right)\right]}{10\left[m(1+2m)z(mz+2)+m^2+4m+2\right]},$$

which happens to fit the observational data perfectly — see Figure 3.

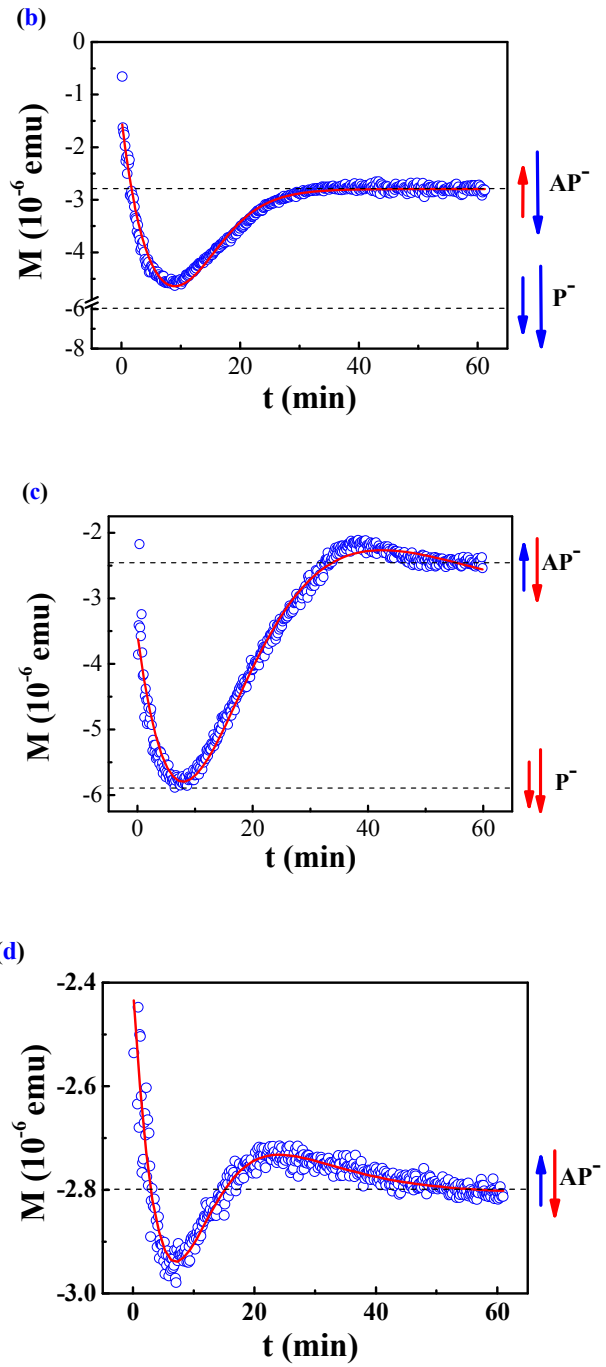

Fig. 3. Magnetic relaxation in Pt(3 nm)/Co(1.05 nm)/Ir(1.5 nm)/Co(0.7 nm)/Pt(3 nm) in the magnetic fields –1360 Oe (b), –1354 Oe (c), –1370 Oe (d) at T = 100 K. The solid lines are exact solutions (16), the blue circles — the results of the observations (see [13])

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