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S. E. Stepanov[1], ***I. I. Tsyganok***[2]

[1, 2] *Financial University under the Government of the Russian Federation*

s. e.stepanov@mail.ru

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On the Tachibana numbers of closed manifolds with pinched negative sectional curvature

Conformal Killing form is a natural generalization of conformal Killing vector field. These forms were extensively studied by many geometricians. These considerations were motivated by existence of various applications for these forms. The vector space of conformal Killing p-forms on an n-dimensional $(1 \leq p \leq n-1)$ closed Riemannian manifold M has a finite dimension $t_p(M)$ named the *Tachibana number*. These numbers are conformal scalar invariant of M and satisfy the duality theorem: $t_p(M) = t_{n-p}(M)$.

In the present article we prove two vanishing theorems. According to the first theorem, there are no nonzero Tachibana numbers on an n-dimensional $(n \geq 4)$ closed Riemannian manifold with pinched negative sectional curvature such that $-1-\delta \leq \sec \leq -1$ for some pinching constant $\delta < (n-1)^{-1}$. From the second theorem we conclude that there are no nonzero Tachibana numbers on a three-dimensional closed Riemannian manifold with negative sectional curvature.

Keywords: Riemannian manifold, conformal Killing — Yano tensor, sectional curvature, vanishing theorem.

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1. Introduction and results

Conformal Killing p-forms or, in other words, *conformal Killing — Yano p-tensors* have been defined on n-dimensional Riemannian manifolds $(1 \leq p \leq n-1)$ more than fifty years ago by S. Tachibana and T. Kashiwada (see [1; 2]) as a natural generalization of conformal Killing vector fields. Since then these forms were extensively studied by many geometricians. These considerations were motivated by existence of various applications for these forms (see, for example, [3; 4]).

The vector space of conformal Killing p-forms on an n-dimensional closed (i.e. compact without boundary) Riemannian manifold (M, g) has a finite dimension $t_p(M)$ named the *Tachibana number* (see [5—7] and etc.). These numbers $t_1(M), \ldots, t_{n-1}(M)$ are conformal scalar invariant of (M, g) and satisfy the duality theorem: $t_p(M) = t_{n-p}(M)$. The theorem is an analog of the well known *Poincaré duality theorem* for the Betti numbers of closed (M, g). Moreover, we proved in [7] that Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$ are equal to zero for an n-dimensional ($n \geq 2$) closed Riemannian manifold (M, g) with negative *curvature operator* $\overset{\circ}{R} : S_0^2 M \to S_0^2 M$ defined on the vector bundle of traceless symmetric tensor fields of order 2 (see [8]). Based on this theorem, we prove here the following theorem.

Theorem 1. *Let (M, g) be an n-dimensional ($n \geq 4$) closed Riemannian manifold with pinched negative sectional curvature. Suppose that $-1-\delta \leq \sec \leq -1$ for an arbitrary $\delta < (n-1)^{-1}$, then Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$ of (M, g) are equal to zero.*

Remark. We recall here that for every $n \geq 4$ there exists a closed n-dimensional Riemannian manifold (M, g) such that its sectional curvatures satisfy the inequalities $-H \le \sec \le -1$ for some pinching constant H, but M does not admit a metric g of constant negative sectional curvature (see [9]).

For the case when $n = 3$ the following theorem is true.

Theorem 2. *If (M, g) is an 3-dimensional closed Riemannian manifold with negative sectional curvature then its Tachibana numbers $t_1(M)$ and $t_2(M)$ are equal to zero.*

2. Proofs of Theorems

Let (M, g) be an n-dimensional $(n \ge 2)$ Riemannian manifold and $S_0^2 M$ is a vector bundle of symmetric traceless 2-forms. For any point $x \in M$ there exists an orthonormal eigen-frame $e_1, \ldots, e_n$ of $T_x M$ such that $\varphi_{ij} = \varphi_x(e_i, e_j) = \mu_i\, \delta_{ij}$ for any $\varphi \in C^\infty(S_0^2 M)$ and for the Kronecker delta δ_{ij}. Then we have the formula (see [10, p. 388])

$$R_{ij}\, \varphi^{ik} \varphi^{j}{}_{k} + R_{ijkl}\, \varphi^{il} \varphi^{jk} = 2 \sum_{i<j} \sec(e_i \wedge e_j)(\mu_i - \mu_j)^2 \qquad (2.1)$$

where $\sec(e_i \wedge e_j) = R(e_i, e_j, e_i, e_j)$ is the *sectional curvature* $\sec \sigma_x$ of (M, g) in the direction of the tangent two-plane section $\sigma_x = \operatorname{span}\{e_i, e_j\}$ at $x \in M$. In turn, the components of the curvature tensor R and the Ricci tensor Ric are denoted by $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ and $R_{ij} = Ric(e_i, e_j)$, respectively.

If in addition, suppose that there is a point $x \in M$ where all sectional curvatures satisfies the double inequality $-A \le \sec \le -B$ for some constant $A > B > 0$, then from equation (2.1) we obtain the inequality (see [11])

$$-n\, A \varphi^{ij} \varphi_{ij} \le R_{ij}\, \varphi^{ik} \varphi^{j}{}_{k} + R_{ijkl}\, \varphi^{il} \varphi^{jk} \le -n\, B\, \varphi^{ij} \varphi_{ij} \qquad (2.2)$$

for an arbitrary nonzero $\varphi_x \in S_0^p\left(T_x^* M\right)$. In addition, we have the double inequality

$$-(n-1)A\varphi^{ij}\varphi_{ij} \leq R_{ij}\,\varphi^{ik}\varphi^{j}_{\ k} \leq -(n-1)B\varphi^{ij}\varphi_{ij} \qquad (2.3)$$

If we suppose that $\|\varphi\|^2 = \varphi^{ij}\varphi_{ij}$, then from (2.2) and (2.3) we obtain the following inequalities

$$\left(-nA + (n-1)B\right)\|\varphi\|^2 \leq g\,(\overset{\circ}{R}(\varphi),\varphi) \leq \left(-nB + (n-1)A\right)\|\varphi\|^2,$$

where $g(\overset{\circ}{R}(\varphi),\varphi) = R_{ijkl}\,\varphi^{il}\varphi^{jk}$ for the curvature operator

$$\overset{\circ}{R} : S_0^2\left(T_x^* M\right) \to S_0^2\left(T_x^* M\right).$$

Let $A = \varepsilon B$, then we have $g(\overset{\circ}{R}(\varphi),\varphi) < 0$ for any nonzero $\varphi \in C^\infty\left(S_0^2 M\right)$ for an arbitrary $0 < \varepsilon < 1 + (n-1)^{-1}$. Theorem 1 is proved.

In dimension three we have (see [12])

$$R_{ijkl} = g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik} - \tfrac{1}{2}\,s\left(g_{ik}g_{jl} - g_{il}g_{jk}\right).$$

On the other hand, we can always diagonalize the Ricci tensor Ric at an arbitrary point $x \in M$, so that $R_{ij} = \lambda_i\,\delta_{ij}$ where $\lambda_1, \lambda_2, \lambda_3$ are its eigenvalue. Then in three dimensions the only nonzero components of the curvature tensor R are components of the form

$$R_{1212} = \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3); \;\; R_{1313} = \tfrac{1}{2}(\lambda_1 + \lambda_3 - \lambda_2);$$
$$R_{2323} = \tfrac{1}{2}(\lambda_2 + \lambda_3 - \lambda_1).$$

Thus, the condition for negative sectional curvature in three dimensions is that each eigenvalue of the Ricci tensor is bigger than sum of other two. Then we have the proposition (compare this statement with Corollary 8.2 of [12]).

Proposition. *In dimension three a metric g has negative sectional curvature if and only if* $Ric > \tfrac{1}{2}s\,g$ *for the scalar curvature* $s = trace_g Ric$.

Then in dimension three we can deduce the inequalities

$$g(\overset{\circ}{R}(\varphi),\varphi) < \tfrac{1}{2} s \|\varphi\|^2 < 0$$

for a nonzero $\varphi \in C^{\infty}\left(S_0^2 M\right)$. In this case, the curvature operator $\overset{\circ}{R} : S_0^2 M \to S_0^2 M$ is negative definite. At the same time, the vanishing theorem from the paper [7] is the stating that Tachibana numbers $t_1(M) = 0, \ldots, t_{n-1}(M) = 0$ for an n-dimensional ($n \geq$) closed Riemannian manifold (M, g) with negative curvature operator $\overset{\circ}{R} : S_0^2 M \to S_0^2 M$. Then in dimension three we have $t_1(M) = t_2(M) = 0$. Thus, Theorem 2 is proved.

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С.Е. Степанов[1], *И.И. Цыганок*[2]

[1,2] *Финансовый университет при Правительстве РФ, Россия*

s.e.stepanov@mail.ru

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Числа Тачибаны замкнутых многообразий с защемленной отрицательной секционной кривизной

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Конформная форма Киллинга является естественным обобщением конформного векторного поля Киллинга. Эти формы широко изучались многими геометрами, что было мотивировано существованием различных приложений для этих форм. Векторное пространство конформных *p*-форм Киллинга имеет на замкнутом *n*-мерном $(1 \le p \le n-1)$ римановом многообразии M конечную размерность $t_p(M)$, называемую числом Тачибана. Эти числа являются конформными скалярными инвариантами многообразия и удовлетворяют теореме двойственности $t_p(M) = t_{n-p}(M)$.

В данной статье мы доказываем две «теоремы исчезновения». В соответствии в первой теоремой не существует ненулевых чисел Тачибаны на *n*-мерном $(n \ge 4)$ замкнутом римановом многообразии с защемленной отрицательной секционной кривизной такой, что $-1-\delta \le \sec \le -1$ для постоянной $\delta < (n-1)^{-1}$. Согласно второй теореме не существует ненулевых чисел Тачибаны на трехмерном замкнутом римановом многообразии с отрицательной секционной кривизной.

Ключевые слова: риманово многообразие, конформный тензор Киллинга — Яно, секционная кривизна, теорема исчезновения.

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