MSC (2020) Primary: 53C43, 58E20; Secondary: 53C20

***S. E. Stepanov*[1], *I. I. Tsyganok*[2]**
[1, 2] *Financial University under the Government of the Russian Federation,*
49, Leningradsky prosp., Moscow, 125993, Russia
[1] s.e.stepanov@mail.ru, [2] i.i.tsyganok@mail.ru
[1] SPIN-код: 8403-0453, AuthorID: 6991
[2] SPIN-код: 4034-2150, AuthorID: 8745
doi: 10.5922/0321-4796-2026-57-1-6

On harmonic sections of tangent bundle equipped with the Sasaki metric

Let (TM, g_S) be the tangent bundle of a complete Riemannian manifold (M, g) equipped with the Sasaki metric g_S and $X: (M, g) \longrightarrow (TM, g_S)$ be a horizontal-harmonic map from (M, g) to (TM, g_S). If $\|X\| \in L^p(M)$ for at least one positive number $p \neq 1$, then $X: (M, g) \longrightarrow (TM, g_S)$ is a harmonic map from (M, g) to (TM, g_S).

Keywords: complete Riemannian manifold, tangent bundle, Sasaki metric, horizontal-harmonic map

1. Introduction and Main Theorem

Let (TM, g_S) be the tangent bundle of a Riemanian manifold (M, g) equipped with the Sasaki metric g_S defined by $g_S\left(X^h, Y^h\right) = g_S(X^v, Y^v) = g(X, Y) \circ \pi$ and $g_S\left(X^h, Y^v\right) = 0$ for all $X, Y \in C^\infty(TM)$, where π is the natural projection from TM onto M, and X^h (resp. X^v) is 1the horizontal (resp. vertical) lift of a vector field X on M to TM (see [1] and [2]).

Submitted on November 22, 2025

The tension field of the map $X: (M, g) \longrightarrow (TM, g_S)$ is given by (see [3, p. 146])

$$\tau(X) = \tau^h(X) + \tau^v(X) = \left(trace_g R(X, \nabla X)\,\cdot\right)^h + \left(trace_g \nabla^2 X\right)^v.$$

Consequently, X is defined as a horizontal-harmonic (respectively, vertical-harmonic) map from (M, g) to (TM, g_S) if and only if $\Delta X := trace_g \nabla^2 X = 0$ (resp, $trace_g R(X, \nabla X)\cdot = 0$) (see also [3, p. 146]). Furthermore, X is a harmonic map from (M, g) to (TM, g_S) if and only if it is a horizontal-harmonic and vertical-harmonic map at the same time (see [4]). If (M, g) is compact and $X: (M, g) \longrightarrow (TM, g_S)$ is a horizontal-harmonic map from (M, g) to (TM, g_S), then X is a harmonic map since from $\Delta X = 0$ implies $\nabla X = 0$ and hence $trace_g R(X, \nabla X)\cdot = 0$. Formulate the main theorem.

Theorem. *Let (TM, g_S) be the tangent bundle of a complete Riemannian manifold (M, g) equipped with the Sasaki metric g_S and $X: (M, g) \longrightarrow (TM, g_S)$ be a horizontal-harmonic map from (M, g) to (TM, g_S). If $\|X\| \in L^p(M)$ for at least one positive number $p \neq 1$, then $X: (M, g) \longrightarrow (TM, g_S)$ is a harmonic map from (M, g) to (TM, g_S).*

2. Proof of the theorem

Let $X: (M, g) \longrightarrow (TM, g_S)$ is a horizontal-harmonic map, then we deduce

$$\frac{1}{2}\Delta \|X\|^2 = g(\Delta X, X) + \|\nabla X\|^2 = \|\nabla X\|^2$$

since $\Delta X = 0$. On the other hand, we have

$$\frac{1}{2}\Delta \|X\|^2 = \|X\|\,\Delta\|X\| + \big\|\nabla\|X\|\big\|^2 \leq$$

$$\leq \|X\|\,\Delta\|X\| + \|\nabla X\|^2$$

by the Kato inequality $\left\|\nabla\|X\|\right\|^2 \leq \|\nabla X\|^2$ that is a classical tool in Riemannian geometry. As a result, we have the following inequality:

$$\|X\|\,\Delta\|X\| \geq 0.$$

In turn, Yau proved the following theorem (see [5]): Let u be a smooth function defined on a complete Riemannian manifold so that $u \geq 0$ and $(p-1)u\,\Delta u \geq 0$ where p is a positive number. Then for $p \neq 1$, either $\int_M u^p dv_g = \infty$ or $u = const$. Therefore, if $\int_M \|X\|^p dv_g < \infty$ for at least one positive number $p \neq 1$, then $\|X\| = const$. In this case, the equation $\nabla X = 0$ holds. As a consequence of the above, we obtain that $trace_g R(X, \nabla X)\cdot = 0$ and hence $X: (M, g) \longrightarrow (TM, g_S)$ is also a vertical-harmonic map. Consequently, X is a harmonic map from (M, g) to (TM, g_S).

References

1. *Sasaki, S.:* On the differential geometry of tangent bundles of Riemannian manifolds. Tohoku Math. J., 10 (3), 338—354 (1958).

2. *Gudmundsson, S.; Kappos, E.:* On the Geometry of Tangent Bundles. Expo. Math., 20, 1—41 (2002).

3. *Gil-Medrano, O.:* Relationship between volume and energy of vector fields. Differ. Geom. Appl., 15, 137—152 (2001).

4. *Ishihara, T.:* Harmonic sections of tangent bundle. J. Math. Tokushima Univ., 13, 23—27 (1979).

5. *Yau, S. T.:* Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. Indiana Univ. J., 25, 659—670 (1976).

For citation: Stepanov, S. E., Tsyganok, I. I.: On harmonic sections of tangent bundle equipped with the Sasaki metric. DGMF, 57 (1), 76—80 (2026). https://doi.org/10.5922/0321-4796-2026-57-1-6.

УДК 514.764

С. Е. Степанов[1], *И. И. Цыганок*[2]
[1, 2] *Финансовый университет при Правительстве РФ, Москва, Россия*
[1] s.e.stepanov@mail.ru, [2] i.i.tsyganok@mail.ru,
doi: 10.5922/0321-4796-2026-57-1-6

О гармонических сечениях касательного расслоения, снабженных метрикой Сасаки

Поступила в редакцию 08.04.2026 г.

Пусть (TM, g_S) будет касательным расслоением полного риманова многообразия (M, g) с метрикой Сасаки g_S и $X: (M, g) \longrightarrow (TM, g_S)$ будет горизонтально-гармоническим отображением. Если $\|X\| \in L^p(M)$ по крайней мере в одной точке для положительного $p \neq 1$, тогда $X: (M, g) \longrightarrow (TM, g_S)$ будет гармоническим отображением.

Ключевые слова: полное риманово многообразие, касательное расслоение, метрика Сасаки, горизонтально-гармоническое отображение

Список литературы

1. *Sasaki S.* On the differential geometry of tangent bundles of Riemannian manifolds // Tohoku Math. J. 1958. Vol. 10, №3. P. 338—354.
2. *Gudmundsson S., Kappos E.* On the Geometry of Tangent Bundles // Expositiones Mathematicae. 2002. Vol. 20. P. 1—41.
3. *Gil-Medrano O.* Relationship between volume and energy of vector fields // Differ. Geom. Appl, 2001. Vol. 15. P. 137—152.
4. *Ishihara T.* Harmonic sections of tangent bundle // J. Math. Tokushima Univ. 1979. Vol. 13. P. 23—27.
5. *Yau S. T.* Some function-theoretic properties of complete Riemannian manifold and their applications to geometry // Indiana Univ. J. 1976. Vol. 25. P. 659—670.

Для цитирования: *Степанов С.Е., Цыганок И.И.* О гармонических сечениях касательного расслоения, снабженных метрикой Сасаки // ДГМФ. 2026. №57 (1). С. 76—80. https://doi.org/10.5922/0321-4796-2026-57-1-6.