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CENTRAL TANGENTIALLY-DEGENERATED HYPERSTRIPS CH^r_m OF RANK r OF PROJECTIVE SPACE P_n

The article is devoted to the investigation of central tangentially-degenerated hyperstrips $CH^r_m \subset P_n$. Representation is brought and existence theorem of the hyperstrip CH^r_m is proved. Generalized normals of the 1st and 2nd genuses of the hyperstrip CH^r_m are introduced. One-parameter bundles of generalized normals of the 1st and 2nd genuses in the neighbourhood of the 3rd order are constructed by interior invariant way and it is shown that these bundles of normals are reciprocal relative to a field of osculating quadrics.

It is shown that in each centre $A \in V_r$ in a differential neighbourhood of the 3rd order one-parameter family of equipping planes of a surface V_r (in the sence of Cartan) is joined by interior invariant way. With the help of focal manifolds, assotioted with the hyperstrip CH^r_m, geometric meanings of some basic quasitensors of the hyperstrip CH^r_m are explained. A structure of construction of a one-parameter bundle of normals of the 2nd genus $N_{m-1}(A)$ of the hyperstrip CH^r_m in the neighbourhood of the 3rd order is carried out.

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GAR-REALIZATIONS OF LINEAR AND UNITARY GROUPS

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1. Abstract

The following question going back to D. Hilbert is called the classical problem of the inverse Galois theory: Given a finite group G, does there exist a Galois extension N/Q with Galois group isomorphic to G?

This problem is not yet completely solved but it could be reduced by Matzat to the question of the existence of **GAR**-realizations of finite simple groups.

A **GA**-realization of a finite, simple nonabelian group H over the field k is a geometric Galois extension $N/k(t)$ with Galois group isomorphic to $Aut(H)$ under the condition that N^H is a rational function field over k. **GA**-realization of H is called a

GAR-realization if every geometric extension R/N^H with $\bar{k}R=N^H\bar{k}$ is a rational function field over k.

If k is Hilbertian (this is the case for $k=Q^{ab}=Q$ and for $k=Q^{ab}$, for example), then for any regular Galois extension $N/k(t)$ there exists by the Hilbert's Irreduzibility-theorem a specialization of $t\to a\in k$ together with a Galois extension over k with isomorphic Galois group.

In this context we should mention a conjecture of Shafarevich, that $Gal(\bar{Q}/Q^{ab})$ is a free profinite group of countable infinite rank. By the result of Matzat this could be proved, if we could find **GAR**-realizations for every simple finite group over Q^{ab}.

The main aim of my **PhD**-thesis was to do this for the unitary group $U_n(p)$ for odd p. One way was to use the character theory of groups of Lie-type, as will be now shown.

For the classical groups over a prime field F_p res. over F_{p^2} in the twisted case are only the groups $L_n(2)$ and $U_n(2)$ are not yet known to have **GAR**-realizations over Q^{ab}.

2.Constructive Galois theory

In this section we recall some basic facts from constructive Galois theory.

Let G be a finite group with class vector $C=(cl(c_1),cl(c_2),cl(c_3))$, $c_i\in G$. Now set

$$\bar{\Sigma}(C)=\{\sigma=(\sigma_1,\sigma_2,\sigma_3)\mid\sigma_i\in cl(c_i),\ \sigma_1\sigma_2\sigma_3=1\},$$

$$\Sigma(C)=\{\sigma\in\bar{\Sigma}(C)\mid<\sigma_1,\sigma_2,\sigma_3>=G\},$$

$$l(C):=\mid\Sigma(C)/Inn(G)\mid\le n(C):=\mid\bar{\Sigma}(C)/Inn(G)\mid.$$

We call n(C) the normalized structure constant of C, which is an lower bound for the number of orbits of G, in its action via conjugation in the components.

There are two useful ways to determine n(C):

We have

$$n(C)=\frac{|G|\,|Z(G)|}{|C_G(c_1)|\,|C_G(c_2)|\,|C_G(c_3)|}\sum_{\chi\in Irr(G)}\frac{\chi(c_1)\chi(c_2)\chi(c_3)}{\chi(1)},$$

and also

$$n(C)=\sum_{[\sigma]\in\bar{\Sigma}(C)/Irr(G)}\frac{|Z(G)|}{|C_G(<\sigma_1,\sigma_2,\sigma_3>)|}.$$

If we set

$$Q_C=Q(\{\chi(c_i)\mid\chi\in Irr(G),\ i=1,2,3\}),$$

we can formulate the Rigidity Theorem.

Rigidity Theorem: Let G be a finite group with a rigid class vector, i.e. with $l(C)=1$, whose center possesses a complement in the normalizer of one of the subgroups $<c_i>$. Then there exists a geometric Galois extension $N/Q_C(t)$ with

$$Gal(N/Q_C(t))\cong G.$$

3.The strategy

Now we will show how we get the solution in the case of unitary group $U_n(p)$ with odd p and odd $n=2m+1$. (The proof for even n and for $L_n(p)$ is similar.)

First we fix the notation:

Let q be a power of p, $G=GU_n(q)$, $\hat{G}=G<\sigma_0>$ the extension of G by the graph automorphism σ_0, $\sigma_0\colon A \to A^{\sigma_0}=A^{-tJ}$, J=antidiag(1,...,1).

The strategy is now clear. First find two elements generating the group $\hat{G}$. Then take a third one and show that the normalized structure constant is 1.

Since there is a classification of the finite primitive irreducible reflection groups over an arbitrary field of odd characteristic due to Wagner, the simplest solution is to choose as c_1 a regular element of the Coxeter-torus, which operates irreducibly and primitively on the natural vector space. For c_2 we choose a quasi-central automorphism, namely

$$c_2=\sigma:=\sigma_0\begin{pmatrix} -E_m & \\ & E_{m+1} \end{pmatrix},\; C_G(c_2)=<-E_n>Sp_{n-1}(q)$$

because $-c_2^2$ is a reflection. Now we can show that the c_2 together with the Coxeter-torus does generate the desired group by excluding all other possibilities. In addition this implies that $n(C)$ has to be an integer.

It is a frequent observation, that the largest contribution to $n(C)$ is due to the l-character. This gives us a hint to take as third element an element so that the product of the orders of the centralizers is close to the group order. This leads us to $c_3=\sigma_0$, $C_G(\sigma_0)=O_n(q)$.

To evaluate the formula for $n(C)$ we make use of the Deligne-Lusztig-theory, which gives us a partition of the set of the irreducible characters in the following way:

For all classes $cl_G(t)$ of semisimple elements there exists a set $\chi(t)=\{\chi_1^t,\ldots,\chi_{r(t)}^t\}$. It holds

$$Irr(G)=\bigcup_{cl_G(t)}\chi(t)$$

A character $\chi\in\chi(t)$ extends to $\hat{G}$ if and only if there is $s=s^{\sigma}\in cl_G(t)$.

Since only characters $\chi\in\chi(t)$ with t in the Coxeter-torus take values unequal 0 on c_1 and the only t with $t\sim t^{\sigma}$ are $\pm E_n$, we have now only to look at the unipotent characters.

The following trick introduced by G. Malle is quite useful:

Since character values are polynomials in q we just look what happens for $q\to\infty$. If the limes exists, it has to be the exact value for every q, because n(C)is an integer and the number of summands in n(C) is independent from q.

We can express the unipotent characters of G, up to sign, with the help of the Deligne-Lusztig-characters R_w and the characters of the Weyl group W=S_n

$$\pm\chi_\phi = R_\phi = \frac{1}{|W|}\sum_{w\in W}\phi(ww_0)R_w$$

where w_0=J and $\phi\in Irr(S_n)$.

Using the known values of the R_ϕ

$$\pm\chi_\phi(c_1)=\pm\phi((n)).$$

Digne and Michel have recently proved that those unipotent characters, which are canonically parametrized by the characters (of the the Weyl group $W(B_{\frac{n-1}{2}}) = S_n^2$, can be written on the outer coset as

$$(\chi_\phi)_{|G.\sigma} = R_\psi = \frac{1}{|W|}\sum_{w\in W(B_m)}\psi(w)R_w\ .$$

The remaining unipotent characters take the value 0 on c_2.

After some easy computation we get

$$R_\psi(c_2)=R_\psi(c_3)=R_\psi^{SO_n}(1)\ .$$

And now we see

$$\partial(\chi(1))>\partial_q(\chi(c_1)\chi(c_2)\chi(c_3)),$$

where ∂_q denotes the degree in q, unless χ=1.

Thus we can prove

Theorem: Let G be the group $U_n(p)$, where n is odd and $2\neq p\in P$. Then G has a **GA**-realization over Q^{ab}.

Ñ. Ð à é ò å ð

ÐÀÖÈÎÍÀËÜÍÛÅ ÃÀÌÇÀ-ÐÅÀËÈÇÀÖÈÈ ËÈÍÅÉÍÎÉ È ÓÍÈÒÀÐÍÎÉ ÃÐÓÏÏ

Ñîãëàñíî ðåçóëüòàòàì Ìàöàòà ðåøåíèå îáðàòíîé çàäà÷è òåîðèè Ãàëóà è äîêàçàòåëüñòâî ãèïîòåçû Øàôàðåâè÷à î òîì, ÷òî ãðóïïà Gal (Q / Q^{ab}) åñòü ñâîáîäíàÿ ïðîêîíå÷íàÿ ãðóïïà ñ÷åòíîãî áåñêîíå÷íîãî ðàíãà, ñâîäÿòñÿ ê îòûñêàíèþ ðàöèîíàëüíûõ Ãàëóà-ðåàëèçàöèé âñåõ ïðîñòûõ êîíå÷íûõ ãðóïï íàä Q^{ab}. Íàñòîÿùàÿ ðàáîòà ïîñâÿùåíà èññëåäîâàíèþ ýòîãî âîïðîñà äëÿ óíèòàðíûõ ãðóïïû $U_k(p)$. Èñïîëüçóÿ êëàññèôèêàöèþ Âàãíåðà êîíå÷íûõ ïðèìèòèâíûõ ãðóïï îòðàæåíèé è òåîðèþ õàðàêòåðîâ ãðóïï òèïà Ëè, óäàëîñü äîêàçàòü ñëåäóþùóþ òåîðåìó:

Åñëè G åñòü óíèòàðíàÿ ãðóïïà U_k (p), ãäå k - íå÷åòíîå, à p - íå÷åòíîå ïðîñòîå ÷èñëî, òî G îáëàäàåò Ãàëóà-ðåàëèçàöèåé íàä Q^{ab}.

Ýòî ïîðîæäàåò ñâÿçü ìåæäó ñòðóêòóðîé ðàñøèðåíèé Ãàëóà ïîëÿ Q^{ab} è êëàññèôèêàöèåé ðèìàíîâûõ ìíîãîîáðàçèé.