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INITIAL PROBLEM FOR HEAT EQUATION WITH MULTISOLITON INHOMOGENEITY AND ONE-LOOP QUANTUM CORRECTIONS[1]

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The generalized zeta-function is built by a dressing method based on the Darboux covariance of the heat equation and used to evaluate the correspondent functional integral in quasiclassical approximation. Quantum corrections to a kink-like solutions of Landau — Ginzburg model are calculated.

Используя одевающий метод, основанный на ковариантности относительно преобразований Дарбу уравнения теплопроводности, мы вычисляем дзета-функцию для дальнейшей оценки функционального интеграла в квазиклассическом приближении. Вычислена квантовая поправка к решению типа «кинк» модели Ландау — Гинзбурга.

Keywords: Darboux transformation, heat conduction equation, kink, functional integral

Ключевые слова: преобразование Дарбу, уравнение теплопроводности, кинк, функциональный интеграл

1. Introduction

In the papers [1—4] quantum corrections to a few classical solutions by means of Riemann zeta-function are calculated. Most interesting of them are the corrections to the kink — the separatrix solution of field ϕ^4 model. The method of [1—3] is rather complicated and it could be useful to simplify it. We use the dressing technique based on classical Darboux transformations (DT) with a new applications to Green function construction [4]. It is the main aim of this note with eventual possibility to generalize the result due to universality of the technique when a link to integrable (soliton, SUSY) systems is established [5]. The suggested approach open new possibilities; for example it allows to show the way to calculate the quantum corrections to Q-balls [6] and periodic solutions of the models. The last problem is posed in the useful review [7].

[1] Это наша последняя совместная работа с безвременно ушедшим профессором С.Б. Лебле. В свое время мы выложили ее в ArXiv и планировали отправить в один из реферируемых журналов. К сожалению, потом мы занялись более «срочными» делами, и работа так и осталась в ArXiv дожидаться своего часа. В память о своем ушедшем старшем товарище я публикую эту статью в «Вестнике» без каких-либо изменений (*А.В. Юров*).

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2. Heat equation Cauchy problem

We will base on the DT-covariance of the heat equation for the function $\rho(\tau, x, y)$:

$$-\rho_\tau + \rho_{xx} + u(x)\rho = 0, \tag{1}$$

that means the form-invariance of (1) with respect to iterated DT, defined by the Wronskian $W[\phi_1, ..., \phi_N]$ of the solutions of (1):

$$\rho \to \rho[N] = \frac{W[\phi_1, ..., \phi_N, \rho]}{W[\phi_1, ..., \phi_N]}, \quad u \to u[N] = u + 2ln_{xx}W[\phi_1, ..., \phi_N]. \tag{2}$$

Consider now a Cauchy problem for the equation (1), where $u(x)$ represents the reflectionless potential in a sense that it could be produced by the DT and the initial condition is

$$\rho(0, x, y) = \delta(x - y). \tag{3}$$

The problem is formulated for a Green function: it is rather general and may be applied as a model of classical diffusion or heat conductivity. We, however, would follow other applications in the theory of quasiclassical quantization, where the function ρ is treated as density matrix whence τ stands for inverse temperature.

The algorithm of such problem solution is the dressing procedure organized by a sequence of DTs defined by (2):

$$\begin{gathered}(\frac{\partial}{\partial x} - \ln_x \phi_1(x, y))\rho_0(0, x, y) = g_1(x, y),\\ (\frac{\partial}{\partial x} - \ln_x \phi_2[1](x, y))g_1(x, y) = g_2(x, y), ...,\\ (\frac{\partial}{\partial x} - \ln_x \phi_k[k-1](x, y))g_{k-1} = g_k x, y,\\ g_N(x, y) = \delta(x, y), 2 \le k \le N\end{gathered} \tag{4}$$

and the following theorem.

Theorem. The function $\rho[N]$ being built by (2) will be a solution of the problem (1, 3) with the potential $u[N]$, if $\rho(\tau, x, y)$ is a solution of the (1) with the initial condition $\rho_0(0, x, y)$.

The result is used when static solutions of ϕ^4 model are quantized by means of Riemann function $\zeta(s)$ $[1-3]$ expressed via the Green functions of the (1) (see also [4]). The one-loop quantum correction to action is evaluated directly as

$$S_q = -\zeta'(0).$$

3. Example of kink

Most popular example of the kink is obtained in this scheme by means of DT over zero seed $u=0$. The solution ρ of (1) with ρ_0 as initial condition for this case is a simple heat equation solution

$$\rho(\tau,x.y)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\rho_0(z,y)\exp[-(x-z)^2/4\tau]dz. \quad (5)$$

The initial condition ρ_0 is evaluated by direct integration in (4):

$$\rho_0(x,y)=\phi_1(x)\begin{cases}\phi_1^{-1}(y), & x>y\\ 0, & x<y\end{cases}. \quad (6)$$

The Green function $\rho[2]$ (density matrix) for the kink solution as the potential is built by the two-fold DT by the Wronskian formula (2) that results in

$$\rho[2](\tau,x,y)=\exp[\frac{-(x-y)^2}{4\tau}]/2\sqrt{\pi\tau}+ \\ +\frac{1}{2}\sum_{m=1}^{2}\rho_m\psi_m(x)\psi_m(y)[Erf[\frac{(x-y+2b_m\tau)}{2\sqrt{\tau}}]-Erf[\frac{(x-y-2b_m\tau)}{2\sqrt{\tau}}], \quad (7)$$

where $b_k=km/\sqrt{2}$, $\rho_k=||\psi||^{-2}$, k=1, 2. After multiplication of the Green function by $\exp[-4m^2\tau]$:

$$\rho\to\rho\exp[-4m^2\tau],$$

the first term of the Green function leads to a divergent integral. This divergency is well-known, its origin is a zero vacuum oscillations. In our approach this fact has transparent explanation, because the divergent term is simply a solution of heat equation with constant coefficients, that appear when the self-action of scalar field is neglected. Such divergence is usually compensated by addition of contra terms of normal order.

Our procedure deletes all ultraviolet divergencies of 1+1 ϕ^4 model including energy of zero oscillations and one-meson states if one evaluates the generalized zeta-function by the formula

$$\zeta_D(s)=M^{2s}\frac{\int_0^{\infty}\gamma(t)t^{s-1}dt}{\Gamma(s)}. \quad (8)$$

$\Gamma(s)$ is the Euler gamma function and M is a mass scale. The function $\gamma(t)$ in the integrand of (8) is expressed via the Green functions $G(x,y,\tau)$ and $G_0(x,y,\tau)$ difference. The result coincides with one from [1—3].

4. Conclustion

As a conclusion let us note that this approach is elaborated in [8] (published in a local conference abstract book) and allows to calculate one-loop corrections to the N-level reflectionless potential and, very similarily, solitons of SG. Some eventual applications are visible in the case studied at [8].

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