MSC (2020): 53C40; 58E12

S. E. Stepanov
Department of Mathematics, Financial University, Russia
s.e.stepanov@mail.ru
SPIN code: 8403-0453, AuthorID: 6991
doi: 10.5922/0321-4796-2026-57-2-8

On stable minimal submanifolds in Riemannian manifolds

In this paper, we study stable minimal submanifolds of Riemannian manifolds. We generalize classical theorems of the theory using a generalized Bochner technique. We consider a complete stable minimal hypersurface M in $(n+1)$-dimensional Riemannian manifold with non-negative Ricci curvature in the direction of the unite normal vector to M is non-negative at every point $x \in M$. It is proved, that if (M, g) is parabolic (in particular, a complete manifold of finite volume) and there exists a function $u(x) \in C^2(M)$ that has no zeros on M and satisfying at each point of M the inequality $\mathcal{L}\, u(x) \leq 0$ with the Jacobi operator $\mathcal{L}$, then (M, g) must be totally geodesic submanifold.

Keywords: complete Riemannian manifold, stable minimal submanifolds, the generalized Bochner technique, superharmonic function, Jacobi operator

1. Introduction

In this paper, we study minimal submanifolds from the perspective of the *generalized Bochner technique*, which are longstanding variational problems in Riemannian geometry. The *generalized Bochner technique* (see, for instance, [1] and [2]) is an important component of geometric analysis (see, for example, [3]).

Submitted on April 8, 2026
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The obtaining rigidity results for minimal and, in particular, stable minimal submanifolds in Riemannian manifolds has been a problem of significant interest in the geometric analysis community. This has been explored from the classical paper [4] to more recent works, such as [5], which also utilized the generalized Bochner technique. As an example, we recall the following theorem (see [1, Theorem 3.1]):

Let (M, g) be an n-dimensional stable manifold minimally immersed in an $(n+1)$-dimensional Riemannian manifold $(\bar{M}, \bar{g})$ with non-negative sectional curvature. If (M, g) is *parabolic*, then it must be totally geodesic and have non-negative sectional curvature. We will continue this research in our paper. Bochner's generalized methodology enables us to obtain new results in this actively studied area of research (see, for example, [6]).

We presented some of our results in reports at the "XIV Belarusian Mathematics Conference" (October 28 — November 1, 2024, Minsk, Belarus).

2. Complete stable minimal submanifolds in Riemannian manifolds

We consider a smooth connected n-dimensional Riemannian manifold (M, g) that is isometrically immersed into an $(n+1)$-dimensional Riemannian manifold $(\bar{M}, \bar{g})$. Such a manifold is called a *hypersurface*.

We denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections on (M, g) and $(\bar{M}, \bar{g})$, respectively. In this case, the *Gauss formula* says that

$$\nabla_X Y = \bar{\nabla}_X Y - g(A_g X, Y)\, N,$$

where X, Y are any vector fields tangent to (M, g), the vector field N is the global unite vector field normal to (M, g) and A_g stands the shape operator of (M, g). The second fundamental form of a hypersurface (M, g) is defined by the identity $(X, Y) := g(A_g X, Y)$, where X, Y are any vector fields tangent to (M, g). Then φ is a symmetric differential 2-form on (M, g).

As we recalled before, stability is related to the non-negativity of the second variation or, equivalently, the non-positivity of the *stability operator* (also called the *Jacobi operator*) on (M, g) (see [5])

$$\mathcal{L} := \Delta + \|\varphi\|_g^2 + \overline{Ric}(N, N),$$

where N is a globally defined unit normal vector field on M, which means $N_x \in T_x^{\perp}M$ and $\bar{g}(N_x, N_x) = 1$ at each point $x \in M$. Therefore, in this case we have

$$\Delta u \leq -\left(\|\varphi\|_g^2 + \overline{Ric}(N_x, N_x)\right) u(x) \tag{1}$$

for any $u(x) \in C^{\infty}(M)$. Then, using (1), we have

$$\frac{1}{2}\Delta u^2 = \|du\|_g^2 + u(x)\,(\Delta u) =$$

$$= \|du\|_g^2 - \left(\|\varphi\|_g^2 + \overline{Ric}(N_x, N_x)\right) u(x)^2 \leq$$

$$\leq -\left(\|\varphi\|_g^2 + \overline{Ric}(N_x, N_x)\right) u(x)^2. \tag{2}$$

Therefore, if $u(x)$ is a function that has no zeros on M and satisfying the inequality $\mathcal{L}\,u(x) \leq 0$, and $\overline{Ric}(N_x, N_x) \geq 0$ at every point $x \in M$, then u^2 is a positive superharmonic function (see: [7, p. 150]). At the same time, a complete manifold (M, g) is *parabolic* if any positive superharmonic function on (M, g) is constant, otherwise, it is considered non-parabolic (see: [7, p. 164]). Therefore, if a hypersurface (M, g) is a complete parabolic Riemannian submanifold of $(\overline{M}, \bar{g})$ and there exists a function $u(x) \in C^2(M)$ that has no zeros on M and satisfying the inequality $\mathcal{L}\,u(x) \leq 0$, then $u(x)$ is a constant and from (1) we deduce that $\varphi = 0$ and $\overline{Ric}(N_x, N_x) = 0$ at every point of M. In this case, (M, g) must be a totally geodesic submanifold of $(\overline{M}, \bar{g})$ and $\overline{Ric}(N_x, N_x)$ also vanishes at every point $x \in M$. This leads to the following straightforward theorem.

Theorem 1. *Let* (M, g) *be a complete stable minimal hypersurface in* $(n+1)$*-dimensional Riemannian manifold* $(\overline{M}, \bar{g})$ *with*

non-negative Ricci curvature $\overline{Ric}$ *in the direction of the unite normal vector* N_x *to* M *is non-negative at every point* $x \in M$. *If* (M, g) *is parabolic* (*in particular, a complete manifold of finite volume*) *and there exists a function* $u(x) \in C^2(M)$ *that has no zeros on* M *and satisfying the inequality* $\mathcal{L}\, u(x) \leq 0$ *at each point of* M, *where* $\mathcal{L}$ *denotes the stability* (*or Jacobi*) *operator*, *then* (M, g) *must be totally geodesic submanifold. Moreover, the Ricci curvature* $\overline{Ric}(N_x, N_x)$ *also vanishes at every point* $x \in M$.

There is a generalization of parabolic manifolds known as *stochastically complete manifolds*. Specifically, any parabolic manifold is stochastically complete, but the converse is not necessarily true. Recall that if we consider a minimal Wiener process on (M, g), that is, a diffusion process generated by the Laplace — Beltrami operator Δ with absorption conditions at infinity, then if the probability of absorption at infinity in a finite amount of time is zero, the manifold (M, g) is said to be stochastically complete (see, for instance, [10]). In conclusion, we can formulate the theorem.

Theorem 2. *Let* (M, g) *be a complete, stable, and stochastically complete minimal hypersurface in a Riemannian manifold* $(\overline{M}, \bar{g})$, *such that the Ricci curvature of* $(\overline{M}, \bar{g})$ *in the direction of the unite normal vector* N_x *to* M *is non-negative at every point* $x \in M$. *If there exists a function* $u(x) \in C^2(M) \cap L^1(M)$ *that has no zeros and satisfies the inequality* $\mathcal{L}\, u(x) \leq 0$ *at each point of* M, *where* $\mathcal{L}$ *denotes the stability* (*or Jacobi*) *operator*, *then* (M, g) *is totally geodesic submanifold.*

Proof. Let (M, g) be a complete, stable, and stochastically complete minimal hypersurface in a Riemannian manifold $(\overline{M}, \bar{g})$, such that the Ricci curvature of $(\overline{M}, \bar{g})$ in the direction of the unite normal vector N_x to M is non-negative at every point $x \in M$.

Next, we recall that if (M, g) a stochastically complete manifold, then every positive superharmonic function $u(x) \in L^1(M)$ is constant (see [10]). Therefore, if a hypersurface (M, g) in $(\overline{M}, \bar{g})$ is a stochastically complete Riemannian manifold, then from (2) for a

function $u(x) \in C^2(M) \cap L^1(M)$ that has no zeros on M, we obtain $\varphi = 0$. Consequently, (M, g) must be a totally geodesic submanifold of $(\bar{M}, \bar{g})$.

Remark. In [9] and [10], with references to [11], respectively, it was claimed that on a complete, non-compact stable minimal hypersurface (M, g) of $(\bar{M}, \bar{g})$, there exists a globally defined positive function $u(x)$ that belongs to the kernel of the Jacobi operator $\mathcal{L}$. In this case, if the Ricci curvature of $(\bar{M}, \bar{g})$ in the direction of the unite normal vector N_x to M is non-negative at every point $x \in M$, then

$$\Delta u = -\left(\|\varphi\|_g^2 + \overline{Ric}(N_x, N_x)\right) u(x) \leq 0.$$

Therefore, $u(x)$ is a positive superharmonic function. This allows us to simplify the requirements of Theorems 1 and 2.

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For citation: Stepanov, S. E.: On stable minimal submanifolds in Riemannian manifolds. DGMF, **57**:2, 79—86 (2026). https://doi.org/10.5922/0321-4796-2026-57-2-8.

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УДК 514.764.274

С. Е. Степанов

Финансовый университет при Правительстве РФ, Москва, Россия

s.e.stepanov@mail.ru

doi: 10.5922/0321-4796-2026-57-2-8

О стабильных минимальных подмногообразиях римановых многообразий

Поступила в редакцию 08.04.2026 г.

В данной работе мы изучаем стабильные минимальные подмногообразия римановых многообразий. Мы обобщаем классические теоремы теории, используя обобщенную технику Бохнера. Рассматривается полная стабильная минимальная гиперповерхность M в $(n+1)$-мерном римановом

многообразии с неотрицательной кривизной Риччи в направлении единичного вектора нормали к M в каждой его точке x. Доказано, что если M параболическое (в частности, полное многообразие конечного объема) и существует функция $u(x) \in C^2(M)$, не имеющая нулей на M и удовлетворяющая в каждой точке многообразия M неравенству $\mathcal{L}\,u(x) \leq 0$ с оператором Якоби $\mathcal{L}$, то M должно быть вполне геодезическим подмногообразием.

Ключевые слова: полное риманово многообразие, устойчивое минимальное подмногообразие, обобщенная техника Бохнера, супергармоническая функция, оператор Якоби

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Для цитирования: *Степанов С.Е.* О стабильных минимальных подмногообразиях римановых многообразий // ДГМФ. 2026. №57 (2). С. 79—86. https://doi.org/10.5922/0321-4796-2026-57-2-8.

ПРЕДСТАВЛЕНО ДЛЯ ВОЗМОЖНОЙ ПУБЛИКАЦИИ В ОТКРЫТОМ ДОСТУПЕ В СООТВЕТСТВИИ С УСЛОВИЯМИ ЛИЦЕНЗИИ CREATIVE COMMONS ATTRIBUTION (CC BY) (HTTP://CREATIVECOMMONS.ORG/LICENSES/BY/4.0/)