УДК 550.338

E. S. Smirnova

ON ASYMPTOTIC EXPRESSION FOR VELOCITY FIELD OF ATMOSPHERE GAS PERTURBED BY 1D ACOUSTIC WAVE

The problem of 1D acoustic wave initiation by a rise of water masses is formulated as a boundary problem at half space $z > 0$. The atmosphere is modeled as a multi-layer gas with an exponential structure of density in each layer. The boundary conditions at $z = 0$ determine the direction of propagation, by link between dynamic variables (pressure, density, and velocity) of the wave. It defines the dynamic projection operators on the subspaces of z-evolution for each layer. The universal formulas for the perturbation of atmospheric variables in an arbitrary layer are derived in frequency and time domains. The explicit expressions for vertical velocity are built by the stationary phase method considering z as large parameter. The resulting formulas can be used to calculate the ionospheric effect by the explicit formula for electron density evolution. This set of explicit relations form a base for a quick algorithm for early diagnostics of tsunami waves.

Задача одномерного возбуждения акустической волны при подъеме водных масс сформулирована как краевая задача в полупространстве. Атмосфера моделируется как многослойный газ с экспоненциальной структурой плотности в каждом слое. Граничные условия определяют направление распространения по связи между динамическими переменными (давление, плотность и скорость) волны. Он определяет операторы динамической проекции на подпространства z-эволюции для каждого слоя. Универсальные формулы для возмущения атмосферных переменных в произвольном слое выводятся в частотной и временной областях. Явные выражения для вертикальной скорости строятся методом стационарной фазы с учетом z как большого параметра. Полученные формулы могут быть использованы для расчета ионосферного эффекта по явной формуле эволюции электронной плотности. Этот набор явных отношений формирует основу для быстрого алгоритма ранней диагностики волн цунами.

Keywords: acoustics, atmosphere, multilayer model, tsunami, boundary regime problem.

Ключевые слова: акустика, атмосфера, многослойная модель, цунами, проблема граничного режима.

Introduction

The detection and prediction of tsunami waves is an urgent task of modern geophysics [1]. Among the various approaches to the problem, a set of investigations aimed at studying the ocean–atmosphere–ionosphere connection is being distinguished. In [2], convincing arguments were presented in favor of the fact that phenomena occurring in the oceans are an important source of waves in the thermosphere. The tsunami wave disturbs in the atmosphere acoustic and internal gravitational waves [4], that affects the total electron concentration [5—7], that proves to be important both in the diagnostics of atmospheric effects and for the detection of tsunami waves at the initial stage [8]. Moreover, observations show that «the first arrival of a transient signal of tsunami-induced waves occurs at a 100-km altitude just 5 min after the tsunami is generated» [9].

Important results were obtained for the exponential atmosphere within the linear theory [4] and for the non-linear generalization in various orders of magnitude of non-linearity [11]. For instance, the dispersion relations were derived, which provided the basis for the developed concepts and practical recommendations for geophysics.

The authors in [10] simulated the ionospheric responses to infrasonic-acoustic waves, using the compressible atmospheric dynamics model. Such, mainly numerical, investigations present different important aspects of the phenomenon in much detail [12]. The theoretical study of the propagation of long acoustic waves in the atmosphere and tsunami detection problem starts from [4; 8; 13]. Taking into account weak heterogeneity for such waves, methods similar to the semi-classical approximation of quantum mechanics [14] can be adopted.

In our work, we study the formulation and analytical solution of the 1D multi-layer problem of purely acoustic perturbations and their ionospheric effect, that excludes internal gravitational waves [15]. This is especially important for waves such as tsunamis with a large amplitude and space scale [3].

The Fourier method is employed to solve the basic equations and deliver the transformation from the time domain to the frequency domain. The solution of the resulting ordinary differential equations transforms to the time domain by inverse Fourier transform to obtain the final integral form. The corresponding integral contains a rapidly-oscillating function, which paves the way for further asymptotic analysis, which we demonstrate here [16].

Basic equations

The proper decomposition of the perturbation into acoustic and entropy modes in a one-dimensional flow, which is studied in this work, is used as the basis for each layer in our multi-layered model. The equations based on the conservation of momentum, energy, and mass determine the behavior of a fluid, as non-dissipative medium [17]. These nonlinear equations model the dynamics of all possible types of motion that can take place in a gas medium [11].

We start with linearized conservation equations in terms of pressure and density variations, $p'(z)$ and $\rho'(z)$ as deviations from hydro-dynamically-stable stationary functions $\overline{p}(z), \overline{\rho}(z)$ which are no longer constants for gas in the gravity field. Consider the problem of the propagation of acoustic waves in an exponentially-stratified atmosphere layer. The pressure and density of the unperturbed atmosphere are described by the law:

$$\overline{p}(z)=p_0\cdot\exp\left(-\frac{z}{H}\right)=\rho_0 gH\cdot\exp\left(-\frac{z}{H}\right);\overline{\rho}(z)=\rho_0\cdot\exp\left(-\frac{z}{H}\right). \qquad (1)$$

Here, $\overline{p}(z)$ is the pressure of the unperturbed atmosphere, p_0 is the pressure at the water–air interface, $\overline{\rho}(z)$ is the density of the unperturbed atmosphere, ρ_0 is the air density at the water–air interface, H is the atmospheric scale height, and z is the current height value.

Let $\vec{V}$ be the velocity vector of the gas flow with the vertical component V_z; $\gamma=C_p/C_v$; C_p,C_v are molar heat capacities at constant pressure and volume correspondingly; $\vec{g}$ is the gravity acceleration field vector, whose components, in the case of vertical gravitational field, are $g_x=0$, $g_y=0$, and $g_z=g$.

Further, in the context of entropy mode introduction, we enter a new variable:

$$\varphi'=p'-\gamma\rho'\frac{\overline{p}}{\overline{\rho}}. \qquad (2)$$

Next, we go to the conventional set of variables:

$$P=p'\cdot\exp\left(\frac{z}{2H}\right),\Phi=\varphi'\cdot\exp\left(\frac{z}{2H}\right),U_z=V_z\cdot\exp\left(-\frac{z}{2H}\right), \qquad (3)$$

where P,Φ,U_z are the new quantities defined in this way and V_z is the vertical velocity of the flow.

Our main intention to simplify the model relates to the plane waves' case. Therefore, we consider the one-dimensional boundary problem for each layer of our model. For such a case, the system of hydro-thermodynamics takes the form:

$$\frac{\partial P}{\partial z}=-\frac{1}{g(\gamma-1)}\frac{\partial^2\Phi}{\partial t^2}+\frac{\gamma-2}{2\gamma H}P+\frac{\Phi}{\gamma H}, \qquad (4)$$

$$\frac{\partial^2\Phi}{\partial z\partial t}=\frac{\gamma-1}{\gamma H}\frac{\partial P}{\partial t}-\frac{\gamma-2}{2\gamma H}\frac{\partial\Phi}{\partial t}, \qquad (5)$$

$$U_z=-\frac{1}{\rho_0 g(\gamma-1)}\frac{\partial\Phi}{\partial t}. \qquad (6)$$

Solution

The statement of the problem of the propagation of the boundary regime and its solution are described in detail in [16]. The resulting recurrence formulas in the time domain are given below:

$$U_n(z,t) = -\frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \frac{C_n(\omega)}{(i\omega - \lambda)} \cdot \exp\left[-ik_n(\omega)(z - h_{n-1})\right] d\omega, \quad (7)$$

$$P_n(z,t) = -\frac{A}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{i\omega t} \frac{2i\rho_0 \left(H\gamma\omega^2 - g\gamma + g\right) C_n(\omega)}{\omega(i\omega - \lambda)\left(\sqrt{\gamma^2 - 4H\gamma\omega^2 / g} + \gamma - 2\right)} \cdot \exp\left[-ik_n(\omega)(z - h_{n-1})\right] d\omega, \quad (8)$$

$$\Phi_n(z,t) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \frac{g\rho_0(\gamma - 1) C_n(\omega)}{i\omega(i\omega - \lambda)} \cdot \exp\left[-ik_n(\omega)(z - h_{n-1})\right] d\omega, \quad (9)$$

here:

$$C_n(\omega) = \prod_{m=1}^{n} \exp\left[-i(h_{m-1} - h_{m-2}) k_{m-1}(\omega)\right] \cdot \exp\left[\frac{h_{m-1} - h_{m-2}}{2H_{m-1}}\right], \quad (10)$$

$$k_n(\omega) = \sqrt{\frac{\omega^2}{H_n g\gamma} - \frac{1}{4H_n^{\,2}}}. \quad (11)$$

Further, the indices will denote the layer number, i.e., H_n is the average atmospheric scale height in the n-layer, h_n is the height of the interface between n- and $n+1$-layers and $n = 0, 1, 2, \ldots$.

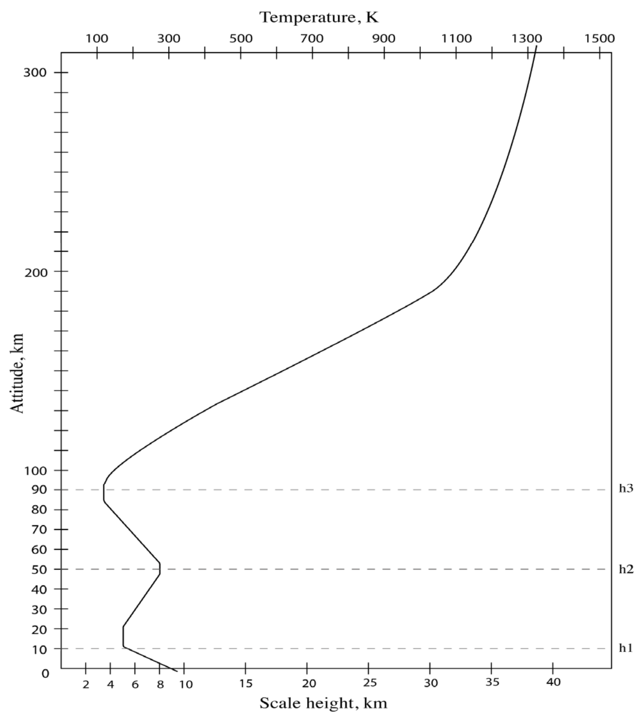

Fig. 1. Multi-layered atmosphere model

We also find critical values for ω when $k_n(\omega) = 0$ for each layer:

$$\omega_{c,n} = \sqrt{\frac{\gamma g}{4H_n}}. \tag{12}$$

Stationary Phase Approximation

The integrand of (7)—(9) contains rapidly-oscillating function $\exp[-ik_n(\omega)(z-h_{n-1})]$ at the range of $(\omega_{c,n},\infty)$ and $z > h_{n-1}$. It allows applying asymptotic representation for the integral by conventional stationary phase approximation via the stationary point frequency ω_{dom}. Generally, having the integral:

$$f(z,t) = \frac{1}{2\pi}\int_{\Re} F(\omega)\cdot\exp\left[-i\left(k(\omega)z - \omega t\right)\right]d\omega \tag{13}$$

the conventional asymptotic expression yields:

$$f(z,t) = \frac{F(\omega_{dom})}{\pi}\sqrt{\frac{2\pi}{z\left|\frac{d^2k}{dz^2}\right|}}\cdot\cos\left(k(\omega_{dom})z - \omega_{dom}t \pm \frac{\pi}{4}\right). \tag{14}$$

Here, the phase term $k(\omega)z - \omega t$ is «stationary» when:

$$\frac{d}{d\omega}\left(k(\omega)z - \omega t\right) \approx 0, \tag{15}$$

$$\frac{dk}{d\omega} = \frac{t}{z}. \tag{16}$$

The root of this equation gives the dominant frequency $\omega_{dom}(z,t)$ for certain *z* and *t*. In the case of (7), for $n = 1$:

$$f(z,t) = U_1(z,t), \tag{17}$$

$$k_1(\omega) = \sqrt{\frac{\omega^2}{H_1 g\gamma} - \frac{1}{4H_1^{\,2}}}, \tag{18}$$

$$F(\omega) = \frac{C_n(\omega)}{(i\omega - \lambda)}. \tag{19}$$

The dominant frequency for (7), found from (4):

$$\omega_{dom,1} = \frac{t\gamma g}{2\sqrt{H\gamma g t^2 - z^2}}. \tag{20}$$

Thus, the velocity in the first layer is represented as:

$$U_1(z,t) = I_c + I_s, \tag{21}$$

where

$$I_c = -\frac{A}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} e^{i\omega t} \cdot \frac{e^{-izk_1(\omega)}}{(i\omega - \lambda)} d\omega. \tag{22}$$

And the stationary phase approximation for $\omega_{dom,1}$ far from ω_c is given by

$$I_s \approx -\frac{A}{2\pi^2} \frac{1}{|i\omega_{dom,1} - \lambda|} \sqrt{\frac{2\pi}{z\left|\frac{d^2k}{dz^2}\right|}} \cos\left(k(\omega_{dom})z - \omega_{dom}t \pm \frac{\pi}{4}\right). \tag{23}$$

We go back to the values from (3):

$$V_1(z,t) = (I_c + I_s)e^{z/2H_1}. \tag{24}$$

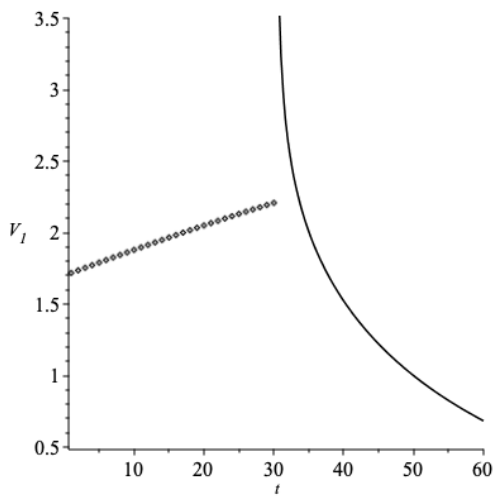

Fig. 2. Depicted V_1 according to the formula (24) for z = 10000 m.
The dots denote the $I_c \exp[z/2H_1]$, calculated numerically.
The solid line indicates $I_s \exp[z/2H_1]$ calculated using
the stationary phase approximation

Second layer

$$V_2(z,t) = -\frac{A}{\sqrt{2\pi}} e^{h_1/2H_1} \int_{-\infty}^{\infty} \frac{1}{(i\omega - \lambda)} \times$$
$$\times \exp\left[i\left(\omega t - h_1\sqrt{\frac{\omega^2}{H_1 g\gamma} - \frac{1}{4H_1^2}} - (z - h_{n-1})\sqrt{\frac{\omega^2}{H_1 g\gamma} - \frac{1}{4H_1^2}}\right)\right] d\omega. \tag{25}$$

The expression for the phase in the second layer:

$$\varphi = \omega t - h_1 \sqrt{\frac{\omega^2}{H_1 g \gamma} - \frac{1}{4H_1^{\,2}}} - \left(z - h_{n-1}\right) \sqrt{\frac{\omega^2}{H_1 g \gamma} - \frac{1}{4H_1^{\,2}}} \tag{26}$$

we rewrite as:

$$\varphi = \omega t - h_1 \left(\sqrt{\frac{\omega^2}{H_1 g \gamma} - \frac{1}{4H_1^{\,2}}} - \frac{z - h_{n-1}}{h_1} \sqrt{\frac{\omega^2}{H_1 g \gamma} - \frac{1}{4H_1^{\,2}}} \right) \tag{27}$$

or

$$\varphi = \omega t - h_1 K(\omega) \tag{28}$$

that allows to apply the general expression (14) directly. The roots of the stationary phase condition $dK/d\omega = t/h_1$ are expressed in rather complicated form that forces us to apply a symbolic computation program.

Analysis of the Effects at Ionospheric Heights. Neutral gas perturbation

To analyze the neutral gas perturbation effect at the altitudes of the ionosphere, it is necessary to return to the physical quantities and then calculate their changes with increasing altitude for each layer:

$$p'_n = P_n \cdot \exp\left(-\frac{z - h_{n-1}}{2H_n} \right), \varphi'_n = \Phi_n \cdot \exp\left(-\frac{z - h_{n-1}}{2H_n} \right),$$
$$U_n = V_n \cdot \exp\left(\frac{z - h_{n-1}}{2H_n} \right). \tag{29}$$

The acoustic wave propagation entering the ionosphere acts on ions. The problem of the AGW ionosphere effect description has been studied for many years [6].

Ionospheric effect

The ionosphere effect, a variation of electron concentration as a function of the vertical coordinate and time, is determined by the vertical component of velocity. In [5], a simple formula for the electron concentration dynamics was derived, and its coordinate and time dependence were calculated as the solution of the diffusion equation, parametrized by the velocity profile as a coefficient. For more details, see [6]. The formula is an expansion of the diffusion equation solution in a series by Whittaker functions with a leading term in the conditions of the considered problem.

Discussion: Comparison

Due to the almost exponential growth of the amplitude with altitude above sea level, even small disturbances at sea level increase at the altitudes of the ionosphere, which gives a gas velocity amplitude over 200 m/s, cf. [9].

This, together with addition information about source localization guarantees the possibility to use the model in tsunami diagnostics and the eventual prognosis of the wave impact at the seashore.

A comparison with the pulse arrival times (t_a) at a prescribed heights, may estimated by the summation of the times $t_n = h_n - h_{n-1}/V_{gn}$, where are the group velocities in n-layer borders. This fact is supported by the measurements results, reported in [9]. More precisely, the arrival time was estimated by the condition $\omega_{dom} \approx \omega_c$ and evaluation of the time delay via the layer group velocities $t_n = 1/k'(\omega_{dom,n})$. For the height of 100 km, it gives the value $t_a = 319s$, the sum of time delays for the first layers, which approximately corresponds to the result of the simulations and experiments given, again in [9].

As for results related to ionosphere perturbations, see again [10]. The simulations show the variation of the electron density at the F-layer of about 8—10 % (Figure 10 of [10]), which is in rather good correspondence with our estimations.

Conclusions

The result of this paper is the final formula for acoustic wave perturbation at ionosphere heights. The expression of the resulting inverse Fourier transform integrals yields an explicit expression for the fluid velocity field by stationary phase method, combined with numerical and symbolic computations. The model of the acoustic wave generated by the earthquake may be built as a similar boundary regime propagation problem.

As a next important result, we do an adaptation of the explicit formula for the infrasonic wave impact to the electron concentration at ionosphere layers.

Finally, combining the analytical formulas creates the basis for the algorithm that can evaluate the ionospheric effect very quickly. Such a model, combined with the event localization model, for tsunami or earthquake diagnostics may be beneficial.

Acknowledgment. Author would like to expresses gratitude to S. Leble for the problem statement and consultations on the content of article and formulation of the results.

This research was funded by FBFR Grant № 18-05-00184 A.

References

1. *Hickey M., Schubert G., Walterscheid R. L.* Propagation of tsunami-driven gravity waves into the thermosphere and ionosphere // J. Geophys. Res. 2009. Vol. 114. doi: 10.1029/2009JA014105.

2. *Godin O. A.* Air-sea interaction and feasibility of tsunami detection in the open ocean // J. Geophys. Res. 2004. Vol. 109. C05002. doi: 10.1029/2003JC002030.

3. *Zabotin N., Godin O., Bullett T.* Oceans are a major source of waves in the thermosphere // J. Geophys. Res. Space Phys. 2016. Vol. 121. P. 3452—3463. doi: 10.1002/2016JA022357.

4. *Hines C. O.* Gravity waves in the atmosphere // Nature. 1972. Vol. 239. P. 73—78.

5. *Karpov I. V., Leble S. B.* The analytical theory of ionospheric IW effect in F2 layer // Geomagn. Aeron.1986. Vol. 26. P. 234—237.

6. *Leble S. B.* Theory of Thermospheric Waves and their Ionospheric Effects // PAGEOPH. 1988. Vol. 127. P. 491—527.

7. *Kherani E. A., Lognonne P., Hebert H. et al.* Modelling of the total electronic content and magnetic field anomalies generated by the 2011 tohokuoki tsunami and associated acoustic-gravity waves // Geophys. J. Int. 2012. Vol. 191. P. 1049—1066. doi: 10.1111/j.1365-246X.2012.05617.x.

8. *Peltier W. R., Hines C. O.* On the possible detection of tsunamis by a monitoring of the ionosphere // J. Geophys. Res. 1976. Vol. 81. P. 1995—2000.

9. *Wei C., Buhler O., Tabak E. G.* Evolution of tsunami-induced internal acoustic–gravity waves // J. Atmos. Sci. 2015. Vol. 72. P. 2303—2317. doi: 10.1175/JAS-D-14-0179.1.

10. *Zettergren M. D., Snively J. B.* Ionospheric response to infrasonic- acoustic waves generated by natural hazard events // J. Geophys. Res. Space Phys. 2015. Vol. 120. P. 8002—8024. doi: 10.1002/2015JA021116.

11. *Perelomova A.* Nonlinear dynamics of vertically propagating acoustic waves in a stratified atmosphere // Acta Acust. United Acust. 1998. Vol. 84. P. 1002—1006.

12. *Occhipinti G., Kherani E., Lognonne A., Geomagnetic P.* Three-dimensional waveform modeling of ionospheric signature induced by the 2004 Sumatra tsunami // Geophys. Res. Lett. 2006. Vol. 33. doi: 10.1029/2006GL026865.

13. *Hines C. O.* Propagation velocities and speeds in ionospheric waves: A review // J. Atmos. Sol. Terr. Phys. 1974. Vol. 36. P. 1179—1204. doi: 10.1016/0021-9169(74) 90107-X.

14. *Babich V. M., Buldyrev V. S., Kuester E. F.* Short-Wavelength Diffraction Theory: Asymptotic Methods. Berlin, 1991.

15. *Leble S., Smirnova E.* Tsunami-Launched Acoustic Wave in the Layered Atmosphere: Explicit Formulas Including Electron Density Disturbances // Atmosphere. 2019. Vol. 10. 629. doi: doi.org/10.3390/atmos10100629.

16. *Dobrokhotov S. Y., Minenkov D. S., Nazaikinskii V. E., Tirozzi B.* Functions of noncommuting operators in an asymptotic problem for a 2D wave equation with variable velocity and localized right-hand side // Oper. Theory Adv. Appl. 2013. Vol. 228. P. 95—126.

17. *Leble S., Perelomova A.* Problem of proper decomposition and initialization of acoustic and entropy modes in a gas affected by the mass force // Appl. Math. Model. 2013. Vol. 13. P. 629—635.

The author

Ekaterina S. Smirnova, Master´s Student, Immanuel Kant Baltic Federal University, Russia.

E-mail: smirnova.ekaterina.serg@gmail.com

Об авторе

Екатерина Сергеевна Смирнова — магистрант, Балтийский федеральный университет им. И. Канта, Россия.

E-mail: smirnova.ekaterina.serg@gmail.com