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# A note on the scalar curvature of a compact Riemannian manifold

In the present paper, we formulate conditions for the constancy of the scalar curvature of an n-dimensional $(n \geq 3)$ compact Riemannian manifold $(M, g)$. In particular, conditions for the constancy of the scalar curvature of $(M, g)$ in the case of the quasi-negative Ricci tensor are found. Conditions are also obtained for a compact Riemannian manifold $(M, g)$ to be an Einstein manifold.

***Keywords*****:** compact Riemannian manifold, scalar curvature, York decomposition, Einstein manifold

## 1. Introduction and notations

We recall the well-known Yamabe problem from 1960: Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ compact Riemannian manifold, then there exists a positive and smooth function $f$ on $M$ such that the Riemannian metric $\bar{g} := f \cdot g$ has the constant scalar curvature $\bar{s}$. In 1984 an affirmative resolution to this problem was provided. Detailed information can be found in the monograph [1, Ch. 4]. In turn, in this article we will formulate conditions for the constancy of the scalar curvature $(M, g)$ and, as a consequence, a criterion for the degeneration of a compact Einstein manifold $(M, g)$ into a Euclidean sphere.

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Let $\nabla$ be the Levi-Civita connection on $(M, g)$ and $S^p M := S^p(T^*M)$ be the vector bundle of symmetric bilinear differential $p$-forms $(p \geq 1)$ on $(M, g)$. We denote by $Ric$ and $s = trace_g Ric$ the Ricci tensor and the scalar curvature of $(M, g)$, respectively. The derivatives of $Ric$ and $s$ are related by the following formula (see [1, p. 35; 43]) $\delta Ric = -\frac{1}{2} ds$, where the differential operator $\delta: C^\infty(S^2 M) \to C^\infty(T^*M)$ is called the divergence (see [1, p. 35]) and defined by the formula $\delta := -trace_g \circ \nabla$. Next define the traceless Ricci tensor $\overset{\circ}{Ric} = Ric - (s/n)g$, then the pointwise orthogonal decomposition $Ric = \overset{\circ}{Ric} + (s/n)g$ holds. In particular, if $\overset{\circ}{Ric} \equiv 0$, then the Ricci tensor $Ric$ satisfies the condition $Ric = (s/n)g$. In this case $(M, g)$ is called the Einstein manifold (see [1, p. 44]). Furthermore, if $n \geq 3$, then $s = const$. An example of an Einstein manifold is the Euclidean sphere $S^n$ equipped with its standard metric.

## 2. The York decomposition for the Ricci tensor

If $(M, g)$ is compact (without boundary), then we can define the $L^2$ inner scalar product of symmetric bilinear differential $p$-forms $\varphi$ and $\phi$ on $(M, g)$ by the formula

$$\langle \varphi, \varphi' \rangle := \int_M g(\varphi, \phi) dvol_g$$

where $dvol_g$ being the volume element of $(M, g)$. We define $\delta^*: C^\infty(T^*M) \to C^\infty(S^2 M)$ the first-order differential operator by the formula $\delta^* \theta := \frac{1}{2} L_\xi g$, where $L_\xi$ is the Lie derivative and $\xi = \theta^\#$ is the vector field dual (by $g$) to the 1-form. Then the differential operator $\delta$ is a formal adjoint operator for $\delta^*$(see [1, p. 35]). In this case, we have $\langle \varphi, \delta^* \theta \rangle = \langle \delta \varphi, \theta \rangle$ for any $\varphi \in C^\infty(S^2 M)$ and $\theta \in C^\infty(T^*M)$.

The following York theorem [2] is a well-known result in Riemannian geometry in the large and it is also included in the monographs (see, e. g., [1, p. 130]).

**Theorem 1**. *For any $n$-dimensional $(n \geq 3)$ compact Riemannian manifold $(M, g)$ the decomposition*

$$C^{\infty}(S^2M) = (\mathrm{Im}\delta^* + C^{\infty}M \cdot g) \oplus \left(\delta^{-1}(0) \cap trace_g^{-1}(0)\right) \quad (1)$$

*holds, where both factors are infinite dimensional and orthogonal to each other with respect to the $L^2$ inner scalar product.*

**Remark.** The second factor $\delta^{-1}(0) \cap trace_g^{-1}(0)$ of (1) is the space of $TT$-tensors on $(M, g)$. At the same time, we recall that a symmetric divergence free and traceless covariant two-tensor is called $TT$-tensor (see, for instance, [3]).

If we suppose $\varphi \in C^{\infty}(S^2M)$, then York $L^2$-orthogonal decomposition formula (1) can be rewritten in the form

$$\varphi = \left(\frac{1}{2}L_{\xi}g + \lambda g\right) + \varphi^{TT} \quad (2)$$

for some $\xi \in C^{\infty}(TM)$, some $TT$-tensor $\varphi^{TT}$ and some scalar function $\lambda \in C^{\infty}(M)$. Applying the operator $trace_g$ to both sides of (2), we obtain $trace_g\varphi = -\delta\theta + n\lambda$, where $\theta$ is the $g$-dual one-form of $\xi$ that means $\theta^{\#} = \xi$ (see [1, p. 30]). In this case, (2) can be rewritten in the form $\mathring{\varphi} = S\theta + \varphi^{TT}$, where

$$\mathring{\varphi} = \varphi - (1/n)\left(trace_g\varphi\right)g$$

is the traceless part of $\varphi$ and

$$S\theta \coloneqq \delta^*\theta + (1/n)\delta\theta g$$

denotes the Cauchy — Ahlfors operator $S: C^{\infty}(T^*M) \to C^{\infty}(S_0^2M)$ actions on the vector space of one-form $C^{\infty}(T^*M)$ and with values in the vector space $C^{\infty}(S_0^2M)$ of symmetric traceless bilinear differential forms (see, e. g., [4]). It's obvious that $S$ annihilates the one-form $\theta$ such that $\theta^{\#} = \xi$ for a *conformal Killing vector* $\xi$ on $(M, g)$, since the conformal Killing vector $\xi$ obeys the equation $\delta^*\theta = -(1/n)\delta\theta \cdot g$ (see [5]). Particular cases of a conformal Killing vector field $\xi$ is a homothetic vector for which $\delta\theta = const$ and a Killing vector, for which $\delta\theta = 0$ (see [5]). Using the above, we can formulate the following corollary.

**Corollary 1**. *For any* $n$*-dimensional* $(n \geq 3)$ *compact Riemannian manifold* $(M, g)$ *the decomposition*

$$C^{\infty}(S_0^2 M) = \mathrm{Im}S \oplus \left(\delta^{-1}(0) \cap trace_g^{-1}(0)\right) \tag{3}$$

*holds, where both terms on the right-hand side of* (3) *are* $L^2$*-orthogonal to each other.*

From the $L^2$-orthogonal decomposition (3) we deduce the $L^2$-orthogonal decomposition for the traceless Ricci tensor

$$\overset{\circ}{Ric} = S\theta + Ric^{TT} \tag{4}$$

for some one-form $\theta \in C^{\infty}(T^*M)$, some $TT$-tensor $Ric^{TT} \in \in C^{\infty}(S^2M)$ and the Cauchy — Ahlfors operator $S$. Therefore, we can formulate the following corollary.

**Corollary 2**. *Let* $\overset{\circ}{Ric}$ *be the traceless Ricci tensor of an* $n$*-dimensional* $(n \geq 3)$ *compact Riemannian manifold* $(M, g)$. *Then the* $L^2$*-orthogonal decomposition* $\overset{\circ}{Ric} = S\theta + Ric^{TT}$ *holds for its traceless Ricci tensor* $\overset{\circ}{Ric}$.

The formal adjoint operator for $S$ is defined by the formula $S^*\omega = 2\delta\omega$ for an arbitrary $\omega \in C^{\infty}(S_0^2M)$ (see [4]). Then the elliptic operator of the second kind $S^*S: C^{\infty}(T^*M) \rightarrow C^{\infty}(T^*M)$ is well known as the *Ahlfors Laplacian* (see also [4]). Note that $kerS^*S = kerS$ since $\langle S^*S\theta, \theta\rangle = \langle S\theta, S\theta\rangle$ for any $\theta \in \in C^{\infty}(T^*M)$. Furthermore, the following equation holds (see [6])

$$S^*S\theta = -\,(n-2)/n\,\cdot ds.$$

Therefore, in general, the scalar curvatures $s$ of $(M, g)$ is constant if and only if the vector field $\xi := \theta^{\#}$ is conformal Killing.In addition, we recall that the kernel of $S$ is trivial if the $Ric$ is quasi-negative (see [5]). Recall that $Ric$ is quasi-negative means that $Ric$ is non-positive everywhere but strictly negative somewhere. The following theorem holds.

**Theorem 2.** *Let* $(M, g)$*be an* $n$*-dimensional* $(n \geq 3)$ *compact Riemannian manifold and* $Ric = \left(\frac{1}{2}L_{\xi}g + \lambda g\right) + Ric^{TT}$ *be the*

*York $L^2$-decomposition of its Ricci tensor $Ric$. Then the scalar curvature $s$ of $(M, g)$ is constant if and only if the vector field $\xi$ is conformal Killing. In particular, if the Ricci tensor $Ric$ of $(M, g)$ is quasi-negative, then the scalar curvature $s$ of $(M, g)$ is constant if and only if the vector field $\xi$ is zero.*

## 3. The York decomposition and Einstein manifolds

Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ compact Einstein manifold such that $Ric = \lambda g$. Then from (4) the equality follows $S\theta + Ric^{TT} = 0$. Therefore, if we applying the operator $\delta$ to both sides of the equality $S\theta + Ric^{TT} = 0$, we obtain $S^{*}S\theta = 0$. As a result from $S\theta + Ric^{TT} = 0$ we deduce that $S\theta = 0$ and $Ric^{TT} = 0$. The opposite is obvious. Using the above, in particular, Theorem 2, we can formulate the following theorem.

**Theorem 3.** *Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ compact Riemannian manifold and*

$$Ric = \left(\frac{1}{2} L_{\xi} g + \lambda g\right) + Ric^{TT}$$

*be the York $L^2$-decomposition of its Ricci tensor $Ric$. Then $(M, g)$ is an Einstein manifold if and only if the vector field $\xi$ is conformal Killing and the $TT$-tensor $Ric^{TT}$ is zero. In particular, if the Ricci tensor $Ric$ of $(M, g)$ is quasi-negative, then $(M, g)$ is an Einstein manifold if and only if the vector field $\xi$ must also be zero as must $Ric^{TT}$.*

According to Theorem 3, we conclude that the definition of an $n$-dimensional $(n \geq 3)$ compact Einstein manifold $(M, g)$ is related to the existence (in general) of a non-zero conformal Killing vector field on $(M, g)$. At the same time, the theorem of Yano and Nagano [7] states that an $n$-dimensional simply connected complete Riemannian manifold $(M, g)$ of positive constant curvature is the only connected complete Einstein manifold admitting a complete conformal vector field $\xi$ which is non-homothetic. Furthermore, $(M, g)$ is conformally diffeomorphic with an $n$-dimensional

Euclidian sphere $S^n$. At the same time, we recall that H. Hopf showed that a compact, simply connected Riemannian manifold with constant sectional curvature 1 is necessarily isometric to the Euclidian sphere $S^n$, equipped with its standard metric (see [8; 9]). Therefore, in the Yano and Nagano theorem, $(M, g)$ must be isometric with the Euclidian sphere $S^n$ if the vector field $\xi$ has a non-constant divergence (see also [5, p. 5]). Using our Theorem 2 and the theorem of Yano and Nagano we can formulate a corollary.

**Corollary 3**. *Let $(M, g)$ be an $n$-dimensional $(n \geq 3)$ simply connected compact Riemannian manifold and let $Ric = \left(\frac{1}{2} L_\xi g + \lambda g\right) + Ric^{TT}$ be the York $L^2$-decomposition of its Ricci tensor, where the vector field $\xi$ has a non-constant divergence. If $(M, g)$ is an Einstein manifold, then it is isometric with an $n$-dimensional Euclidian sphere $S^n$.*

**Remark.** An $n$-dimensional ($n \geq 2$) Riemannian manifold $(M, g)$ is a Ricci almost soliton if and only if the identity $Ric^{TT} = 0$ holds in the orthogonal decomposition of the Ricci tensor (4) (see [6]).

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# Заметка о скалярной кривизне компактного риманова многообразия

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В данной статье формулируются необходимые и достаточные условия постоянства скалярной кривизны n-мерного ($n \geq 3$) компактного риманова многообразия $(M, g)$. В частности, найдены условия постоянства скалярной кривизны компактного риманова многообразия в случае квазиотрицательного тензора Риччи. Также получены условия того, что компактное риманово многообразие $(M, g)$ является многообразием Эйнштейна.

*Ключевые слова*: компактное риманово многообразие, скалярная кривизна, разложение Йорка, многообразие Эйнштейна

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