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V. A. Yurov

GENERALIZED EVANS FUNCTION FOR CONTINUOUS SPECTRUM

Решается задача определения функции ЕН (λ) такой, что если { λ_n } – точки непрерывного спектра оператора Н и $\lambda \neq \lambda_n$, то ЕН (λ) определена и не равна нулю.

The task is to define a function EH(λ), such that if { λ_n } are the points of the continuous spectrum of operator H and $\lambda \neq \lambda_n$, then EH(λ) is defined and is non-zero.

Ключевые слова: функция Эванса, дзета-функция, непрерывный спектр.

Key words: Evans function, Zeta function, continuous spectrum.

Introduction

This article is dedicated to the task of defining a function $E_H(\lambda)$, associated with a differential operator *H*, that is sensitive to the spectrum of this operator. If we restrict our attention to the discrete spectrum σ_{disc} (*H*) and require that equality $E_H(\lambda)$ = 0 should only de satisfied for $\lambda \in$ and require that equality $E_H(\lambda)$ = 0 σ_{disc} (*H*), we arrive to a familiar object, called the Evans function. Originally introduced by J. W. Evans [1] in attempt to analyze the stability of nerve pulses in the Fitzhugh — Nagumo system, it has since became pretty much a staple tool in mathematical physics (cf., [2—10]). However, it's usefulness proves to be severely limited when one takes the continuous spectrum (σ_{cont}) into the picture; it is not uncommon to end up with the Evans function being exactly equal to zero if some special properties of continuous spectrum are not satisfied. In many parts this difficulty stems from the way the Evans function is commonly defined as a Wronskian of two solutions possessing the specific asymptotes at +∞ and -∞ respectively.

The purpose of this article therefore lies in the attempt to make a step in a different direction and take a look at different possibilities for the Evans function that would not heavily depend on the position of continuous spectrum, but would instead provide us with information about σ_{cont} itself. Namely, we are trying to define such an object that if $\{\lambda_n\}$ are the points of the continuous spectrum of operator *H* and $\lambda \neq \lambda_n$, then $E_H(\lambda)$ is defined and is non-zero. The requirement for this object to be equal to zero for the points belonging to the point spectrum remains similar to the requirements for the standard Evans function.

Main definitions

Let $H\psi_n = \lambda_n\psi_n$, and let $\{\lambda_n\}$ denote the set of all discrete and continuous spectra points. Let λ be an arbitrary real parameter and define the operator: $A = H - \lambda$. Obviously, $A\psi_n = (\lambda_n - \lambda)\ \psi_n$, i. e. the eigenvalues of operator A have the form $a_n = \lambda_n - \lambda$. If the spectrum of A is the discrete one,

$$\det A = \det(H - l) = \prod_n a_n = \prod_n (l_n - l).$$

Note, that Evans function is usually defined on the class of rapidly decreasing potentials with the aid of a certain integral representation. If, however, we turn our attention to the class of reflectionless potentials, it is possible to show via Darboux transformations (cf. [2] for example), that the Evans function is a legible candidate for the factorization presented above. The rigorous proof of this important fact will be a subject of a separate article..

Now let us introduce $\varsigma_A(s) = \sum_n a_n^{-s}$. Using the definition of Γ-function we can write (Among other applications, this representation proves to be extremely handy in calculating required by the one-loop quantum correction. [3]):

$$\varsigma_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt \quad t^{s-1}\sum_n e^{-a_n t}, \tag{1}$$

when $\lambda < \lambda_n$, and

$$\varsigma_A(s) = \frac{(-1)^s}{\Gamma(s)}\int_0^\infty dt \quad t^{s-1}\sum_n e^{-|a_n|t}, \text{ for } \lambda > \lambda_n .$$

Next, introduce the Green functions:

$$G_-(t;x,y) = \sum_n e^{-a_n t}\ \Psi_n^*(y)\,\Psi_n(x); \quad \lambda < \lambda_n,$$
$$G_+(t;x,y) = \sum_n e^{a_n t}\ \Psi_n^*(y)\,\Psi_n(x); \quad \lambda > \lambda_n.$$

These functions satisfy the equations

$$\frac{\partial G_-}{\partial t} + AG_- = 0, \frac{\partial G_+}{\partial t} - AG_+ = 0, \tag{2}$$

with initial conditions

$$G\pm(0;x,y) = \delta^{(D)}(x - y). \tag{3}$$

On a final remark, note that

$$\sum_n e^{-a_n t} = \int d^D x\, G_-(t;x,x), \sum_n e^{a_n t} = \int d^D x\, G_+(t;x,x). \tag{4}$$

And, therefore

$$\varsigma_A(s) = \begin{cases} \dfrac{1}{\Gamma(s)}\displaystyle\int_0^\infty dt \quad t^{s-1}\int d^D x \quad G_-(t;x,x), \ \lambda < \lambda_n, \\ \dfrac{(-1)^s}{\Gamma(s)}\displaystyle\int_0^\infty dt \quad t^{s-1}\int d^D x \quad G_+(t;x,x), \ \lambda > \lambda_n. \end{cases}$$

Continuous spectrum (a free particle case)

Let $H = -\Delta = -\frac{\partial^2}{\partial x^\mu \partial x^\mu}$ with μ = 1,..., D. Then (2) transforms into

$$\frac{\partial G_-}{\partial t} - \left(\Delta_D + \lambda\right) G_- = 0. \tag{5}$$

The Fourier integral of G_- has the form

$$G_-(t;x,y) = \int \frac{d^D k}{(2\pi)^{D/2}} e^{ik_\mu \eta_\mu} g(t),$$

where $\eta\mu = x\mu - y\mu$. Solution of (5), satisfying (3) is

$$G_-(t;x,y) = \int \frac{d^D k}{(2\pi)^D} e^{ik_\mu \eta_\mu + (\lambda - k_\mu^2)t},$$

or, in other words,

$$G_-(t;x,y) = \frac{e^{\lambda t}}{2^D (\pi t)^{D/2}} e^{-\eta_\mu^3/(4t)}. \tag{6}$$

Alternatively, one can deduce

$$G_+(t;x,y) = \int \frac{d^D k}{(2\pi)^D} e^{ik_\mu \eta_\mu + (\lambda - k_\mu^2)t}.$$

However, this integral diverges, thus the function $G_+(t;x,y)$ doesn't exist.

Upon substitution of (6) into (1) and (4) we get

$$\varsigma_\lambda(s) = \frac{V_D}{2^D \pi^{D/2} \Gamma(s)} \int_0^{+\infty} dt \;\; t^{s-1-p/2} e^{\lambda t}, \tag{7}$$

where $V_D = \int d^D x = +\infty$ — is a total volume of the configuration space. In order to incorporate this quantity into our considerations and receive a meaningful answer we will invoke a usual trick: assume it to be of finite value now and will take a limit $V_D \to \infty$ at the very end.

Let us assume that $\lambda < 0$, i. e. that $\lambda = -\mu$. Then

$$\varsigma_\lambda(s) = \frac{V_D}{2^D \pi^{D/2} \mu^{2s-D}} \frac{\Gamma(s - D/2)}{\Gamma(s)}.$$

Now it is natural to introduce a function that will serve as a *density* of $\varsigma_\lambda(s)$ and as such, will not depend on the configurational volume V_D :

$$\varsigma_\lambda^{(D)}(s) = \frac{d\varsigma_\lambda(s)}{dV_D} = \frac{1}{2^D \pi^{D/2} \mu^{2s-D}} \frac{\Gamma(s - D/2)}{\Gamma(s)}.$$

Finally, rewrite the determinant of $H - \lambda$ as:

$$det(H - \lambda) = \left(e^{-\varsigma_\lambda{}'(0)}\right)^{V_D},$$

and one can introduce what we'll call the «geometric» Evans function

$$\varepsilon_D(\lambda) = e^{-\varsigma_\lambda'(0)}.$$

Function ε_D is connected to «canonic» Evans function via

$$E_D(\lambda) = \varepsilon_D(\lambda)^{V_D}.$$

Before we move further on let us show a couple of examples of the «geometric» Evans function:

$$\varepsilon_1(\lambda) = e^{\mu}, \varepsilon_2(\lambda) = e^{-\mu^2\left(\ln \mu^2 - 1\right)/(4\pi)}, \varepsilon_3(\lambda) = e^{-\mu^6/(6\pi)}, \varepsilon_4(\lambda) = e^{\mu^2\left(2\ln \mu^2 - 3\right)/\left(64\pi^2\right)},$$
$$\varepsilon_5(\lambda) = e^{\mu^5/\left(60\pi^2\right)}, \varepsilon_6(\lambda) = e^{-\mu^6\left(6\ln \mu^2 - 11\right)/\left(2304\pi^3\right)}, \varepsilon_7(\lambda) = e^{-\mu^7/\left(840\pi^3\right)}$$

(we remind here, that $\lambda = -\mu^2$).

It is also rather helpful to keep in mind that

$$\frac{d}{ds}\varsigma_\lambda^{(D)}(s) = -\frac{\ln \mu^2 - \Psi(s - D/2) + \Psi(s)}{2^D \pi^{D/2}\left(\mu^2\right)^{s-D/2}} \frac{\Gamma(s - D/2)}{\Gamma(s)}.$$

The geometric Evans function $\varepsilon_D(\lambda)$ is defined for $\lambda < 0$ and $\lambda < \lambda_n$.

Note, that the case $\lambda > \lambda_n$ doesn't allow for a Green function (i.e. G_+ doesn't exist). The case $\lambda = \lambda_n$ cannot be analyzed since in this case the corresponding equation turns into an identity. Finally, when $\lambda = 0$, $s \to 0$ $s \to$ 0, the equation (7) diverges at the lower limit, hence ζ-function doesn't exist either.

In conclusion: $\varepsilon_D(\lambda)$ is defined and non-zero only for: $\lambda < 0, \lambda > \lambda_n$, and, therefore, the continuous spectrum exist at $\lambda_n > 0$. To put it in other terms, $\varepsilon_D(\lambda)$ is defined and finite only when $\lambda_n \in$ continuous spectrum (i.e., $\lambda_n > 0$) and when $\lambda = \lambda_n$ for any λ_n (i.e. for $\lambda < 0$).

Thus, $\varepsilon_D(\lambda)$ satisfies all the conditions specified in the statement of the problem.

References

1. *Evans J.W.* Nerve axon equations. Stability of the nerve impulse // Indiana Univ. Math. J. 1972/73. 22. P. 577—593.
2. *Leble S.B., Salle M.A., Yurov A.V.* Darboux transforms for Davey — Stewartson equations and solitons in multidimensions // Inverse Problems. 1992. Vol. 8, №2. P. 207—218.
3. *Bordag M., Yurov A.* Spontaneous symmetry breaking and reflectionless scattering data // Physical Review D — Particles, Fields, Gravitation and Cosmology. 2003. Vol. 67, №2.
4. *Jones C.K.R.T.* Stability of the traveling wave solution of the Fitzhugh — Nagumo system // Transactions of the AMS. 1984. 286 (2). P. 431—469.
5. *Alexander J., Gardner R., Jones C.* A topological invariant arising in the stability analysis of traveling waves // J. Reine Angew. Math. 1990. 410. P. 167—212.
6. *Pego R.L., Weinstein M.I.* Eigenvalues, and instabilities of solitary waves // Philos. Trans. Roy. Soc. London Ser. A, 1992. 340 (1656). P. 47—94.
7. *Gardner R.A., Zumbrun K.* The gap lemma and geometric criteria for instability of viscous shock profiles // Comm. Pure Appl. Math. 1998. 51 (7). P. 797—855.
8. *Sandstede B.* Stability of travelling waves // Handbook of dynamical systems. Amsterdam, 2002. Vol. 2. P. 983—1055.

9. *Benzoni-Gavage S.* Stability of semi-discrete shock profiles by means of an Evans function in infinite dimensions // J. Dynam. Differential Equations. 2002. 14(3). P. 613—674.

10. *Coombes S., Owen M.R.* Evans functions for integral neural field equations with Heaviside firing rate function // SIAM J. Appl. Dyn. Syst. 2004. 3 (4). P. 574—600.

11. *Gesztesy F., Latushkin Y., Makarov K.A.* Evans functions, Jost functions, and Fredholm determinants // Arch. Ration. Mech. Anal. 2007. 186 (3). P. 361—421.

12. *Deng J., Nii S.* An infinite-dimensional Evans function theory for elliptic boundary value problems // J. Differential Equations. 2008. 244 (4). P. 753—765.

Об авторе

Валериан Артемович Юров — асп., Колумбийский университет, Миссури, США, e-mail: vayt37@mail.missouri.edu

Author

Valerian Yurov — PhD Student, Columbia University, Missouri, USA, e-mail: vayt37@mail.missouri.edu

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А. А. Юрова

ДИНАМИКА ЛОКАЛИЗОВАННОГО ИМПУЛЬСА, ОПИСЫВАЕМОГО УРАВНЕНИЕМ ДЭВИ — СТЮАРТСОНА II

Изучен класс точных локализованных решений уравнения Дэви — Стюартсона II типа; показано, что с течением времени такие решения теряют пространственную локализацию с характерным временным масштабом, совпадающим с характерным пространственным масштабом начальной локализации. Потеря локализации выражается в появлении резонансных пиков, число которых определяется характером поведения опорной функции на бесконечности. В частности, экспоненциально локализованные возмущения распадаются на бесконечное число резонансов.

A class of spatially localized solutions of Davey — Stewartson II equation is examined; it is shown that such solutions tend to lose the locality properties with time scale corresponding to a characteristic space scale of initial localization. The locality loss manifests itself with emergence of resonance spikes, whose total number is determined by the asymptotic behavior of support function on infinity. In particular, the exponentially localized perturbations split into an infinite number of the resonances.

Ключевые слова: уравнение Дэви — Стюартсона, преобразование Дарбу, солитоны, интегрируемые системы, пары Лакса.

Key words: Davey — Stewartson equation, Darboux transform, solitons, integrable systems, Lax pairs.