MSC 2010: 53C20

S. E. Stepanov[1], **I. I. Tsyganok**[2]
[1, 2] *Financial University under the Government of the Russian Federation, 49, Leningradsky Prosp., Moscow, 125993, Russia*
[1] s.e.stepanov@mail.ru, [2] i.i.tsyganok@mail.ru
doi: 10.5922/0321-4796-2025-56-10

York decompositions for the Codazzi, Killing and Ricci tensors

The York decomposition of the space of symmetric two-tensors originated in theoretical physics and has found applications in Riemannian geometry, as illustrated by its use in Besse's famous monograph on Einstein manifolds. In this paper, we derive York decompositions for Codazzi, Killing and Rucci tensors on a closed Riemannian manifold. In particular, we derive the York decompositions for the Codazzi, Killing and Ricci tensors with constant trace.

Keywords: closed Riemannian manifold, York decomposition, Codazzi tensor, Killing tensor

1. Introduction

In the present paper we consider a closed (i. e., compact and without boundary) Riemannian manifold (M, g) of dimension $n \geq 2$. We denote by $S^pM := S^pT^*M$ the vector bundle of covariant symmetric p-tensors $(p \geq 1)$ on (M, g) and define the L^2 *global scalar product* of two covariant symmetric p-tensors φ and φ' on (M, g) by the formula

$$\langle \varphi, \varphi' \rangle := \int_M g\,(\varphi, \varphi')dvol_g < +\infty$$

Submitted on November 14, 2024

where $dvol_g$ is the volume element of (M, g). Also $\delta^*: C^\infty(TM) \rightarrow$ $\rightarrow C^\infty(S^2M)$ will be the first-order differential operator defined by the formula $\delta^*\theta := \frac{1}{2} L_\xi g$ for some smooth vector field ξ and it's g-dual one-form θ(see [1, p. 117, 514]). At the same time, we denote by the formula $\delta: C^\infty(S^2M) \rightarrow C^\infty(TM)$ the formal adjoint operator for δ^* which is called the *divergence of symmetric two-tensors*. In this case, we have $\langle \varphi, \delta^*\theta \rangle = \langle \delta\varphi, \theta \rangle$ for any $\varphi \in$ $\in C^\infty(S^2M)$ and $\theta \in C^\infty(T^*M)$.

We recall, that $\varphi \in C^\infty(S^2M)$ is called the *Codazzi tensor* if it satisfies the differential equation (see [1, p. 434; 2, p. 350])

$$(\nabla_X\varphi)(Y, Z) = (\nabla_Y\varphi)(X, Z) \quad (1)$$

for arbitrary $X, Y, Z \in TM$. Such tensors arise naturally in the study of Riemannian manifolds with harmonic curvature or harmonic Weyl tensor (see [1, p. 435]). For example, any Codazzi tensor φ on (M, g) with constant curvature C has the local expression (see [1, p. 436])

$$\varphi = Hess\,(f) + C \cdot f \cdot g$$

for the C^2 — function f on (M, g).

Let us also recall that a symmetric, divergence-free and traceless covariant two-tensor is called a *TT-tensor* (see, for instance, [3]). Any TT-tensor is denoted by φ^{TT} (see [3]). In this case, φ^{TT} satisfies the equations $trace_g\varphi^{TT} = 0$ and $\delta\,\varphi^{TT} = 0$. As a consequence of a result of Bourguignon — Ebin — Marsden (see [1, p. 132] and [4]) the space of TT-tensors is an infinite-dimensional R-vector space on any closed Riemannian manifold (M, g). Such tensors are of fundamental importance in stability analysis in General Relativity (see, for instance, [5; 7]) and in Riemannian geometry (see, e. g., [1, p. 346—347; 4; 8]).

Now, we are ready to formulate our first result.

Theorem 1. *Let* (M, g) *be an* n*-dimensional* $(n \geq 3)$ *closed Riemannian manifold. Then any Codazzi tensor* $\varphi \in C^\infty(S^2M)$ *has the* L^2*-orthogonal decomposition*

$$\varphi = \left(\frac{1}{2}L_\xi g + \lambda\, g\right) + \varphi^{TT} \qquad (2)$$

for some vector field $\xi \in C^\infty(TM)$, *some TT-tensor* $\varphi^{TT} \in$ $\in C^\infty(S^2M)$ *and some scalar function* $\lambda \in C^\infty(M)$. *Furthermore, if the inequality* $\int_M L_\xi\left(trace_g\varphi\right) dvol_g \geq 0$ *holds, then this decomposition can be rewritten as*

$$\varphi = \frac{1}{n}\left(trace_g\varphi\right)g + \varphi^{TT}. \qquad (3)$$

Moreover, if $trace_g\varphi = const$, *then* φ^{TT} *is also a Codazzi tensor.*

Remark. Our theorem generalizes the result of Simons (see Theorem 5.4.1 and Theorem 5.4.2 from [9]): If φ is a traceless Codazzi tensor on a closed Riemannian manifold (M, g), then $\varphi = \lambda\, g + H$, where λ is a constant and H is another traceless Codazzi tensor.

We recall, that $\varphi \in C^\infty(S^2M)$ is called the *Killing tensor* if it satisfies the differential equation (see, for instance, [10])

$$(\nabla_X\varphi)(Y,Z) + (\nabla_Y\varphi)(Z,X) + (\nabla_Z\varphi)(X,Y) = 0 \qquad (4)$$

for arbitrary $X, Y, Z \in TM$. In mathematics, a Killing tensor is a generalization of a *Killing vector*, for symmetric tensor fields. It is a concept in Riemannian and pseudo-Riemannian geometry, and is mainly used in the theory of general relativity. For example, if (M, g) is a Riemannian manifold of constant curvature, then any Killing tensorφ on (M, g) has the local expression (see [11; 12])

$$\varphi_{ij} = e^{2\omega}\left(A_{ijkl}\, x^k x^l + B_{ijk}x^l + C_{ij}\right)$$

for $\omega = (n+1)^{-1}\ln(\det g)$ with respect to a local coordinate system$\{x^1, \dots, x^n\}$of(M, g). The coefficients A_{ijkl}, B_{ijk}, and C_{ij} are constant and symmetric with respect to the first two subscripts and

$$A_{ijkl} + A_{jkil} + A_{kijl} = 0; B_{ijk} + B_{jki} + B_{kij} = 0$$

for $i, j, k, l = 1, \dots, n$.

Now, we are ready to formulate our second result.

Theorem 2. *Let* (M, g) *be an* n*-dimensional* $(n \geq 3)$ *closed Riemannian manifold. Then any Killing tensor* $\varphi \in C^{\infty}(S^2M)$ *has the* L^2*-orthogonal decomposition*

$$\varphi = \left(\frac{1}{2} L_{\xi} g + \lambda\, g\right) + \varphi^{TT}$$

for some vector field $\xi \in C^{\infty}(TM)$, *someTT-tensor* $\varphi^{TT} \in C^{\infty}(S^2M)$ *and some scalar function* $\lambda \in C^{\infty}(M)$. *Furthermore, if the equality* $\int_M L_{\xi}\left(trace_g \varphi\right) dvol_g \leq 0$ *holds, then this decomposition can be rewritten as:*

$$\varphi = \frac{1}{n}\left(trace_g \varphi\right) g + \varphi^{TT}. \qquad (5)$$

Moreover, if $trace_g \varphi = const$, *then* φ^{TT} *is also a Killing tensor.*

The Ricci tensor $Ric$is an important mathematical object used in differential geometry, and it also appears frequently in general relativity (see [1]). It has the local expression $Ric = (s/n)\, g + \overset{\circ}{Ric}$, where Ric is its traceless part. Our next theorem is especially important.

Theorem 3. *Let* (M, g) *be an* n*-dimensional* $(n \geq 3)$ *closed Riemannian manifold. Then the traceless part* $\overset{\circ}{Ric}$ *of the Ricci tensor Ric of* (M, g) *has the* L^2*-orthogonal decomposition*

$$\overset{\circ}{Ric} = S\theta + \varphi^{TT}$$

for the Cauchy — Ahlfors operator $S\theta$, *some one-form* $\theta \in C^{\infty}(T^*M)$ *and some TT-tensor* $\varphi^{TT} \in C^{\infty}(S^2M)$. *Furthermore, if the inequality* $\int_M \left(L_{\xi}\, s\right) dvol_g \geq 0$ *holds, then this decomposition can be rewritten as*

$$Ric = \frac{1}{n}\, s\, g + \varphi^{TT},$$

where s *is constant.*

2. Proofs of theorems

For any n-dimensional $(n \geq 3)$ closed Riemannian manifold (M, g), the algebraic sum $\mathrm{Im}\delta^* + C^{\infty}(M) \cdot g$ is closed in S^2M, and we have the *York decomposition* (see [6; 7, p. 24—25])

$$S^2M = (\mathrm{Im}\delta^* + C^\infty M \cdot g) \oplus \left(\delta^{-1}(0) \cap \mathrm{trace}_g^{-1}(0)\right) \quad (6)$$

where both factors on the right side are infinite-dimensional and orthogonal to each other with respect to the L^2 global scalar product (see [1, p. 130]). It's obvious that the second factor $\delta^{-1}(0) \cap \mathrm{trace}_g^{-1}(0)$ of (6) is the space of *TT*-tensors. Therefore, in particular, we have the L^2orthogonal decomposition (2) for any Codazzi and Killing tensors, respectively.

Let us consider equation (4) of a symmetric Killing tensor φ. From (4) we obtain

$$\delta\,\varphi = \frac{1}{2}\,d\big(trace_g\varphi\big). \quad (7)$$

At the same time, from (5) we can conclude that $trace_g\varphi = \delta\,\theta + n\lambda$, where $\delta\,\theta = -\,div\,\xi$ for $\theta^\# = \xi$. In this case, if $\mathring{\varphi}$ denotes the traceless part of φ, then

$$\mathring{\varphi} = \varphi + \frac{1}{n}(\delta\,\theta - n\lambda)g = \left(\frac{1}{2}L_\xi g + \frac{1}{n}\delta\,\theta\,g\right) + \varphi^{TT}$$

and hence

$$\mathring{\varphi} = 2S\theta + \varphi^{TT}, \quad (8)$$

where $S\theta = L_\xi g + 2/n\ \delta\,\theta\,g$ is the *Cauchy — Ahlfors operator*. Next, applying δ to both sides of (8), we obtain

$$\delta\mathring{\varphi} = S^*S\theta, \quad (9)$$

for the *Ahlfors Laplacian* S^*S for $S^* := 2\delta$ (see details in [13]). Using (7), equation (9) can be rewritten in the form

$$\delta\mathring{\varphi} = \frac{n+2}{n}\,d\big(trace_g\varphi\big). \quad (10)$$

From (9) and (10) we deduce the following integral formula

$$\langle S\theta, S\theta\rangle = \frac{n+2}{n}\int_M L_\xi\big(trace_g\varphi\big)dvol_g. \quad (11)$$

If we assume that $\int_M L_\xi\big(trace_g\varphi\big)dvol_g \le 0$, then from (11) we obtain that $S\theta = 0$ and $\int_M L_\xi\big(trace_g\varphi\big)dvol_g = 0$. In this case, we have $\overset{\circ}{\varphi} = \varphi^{TT}$ and hence (5) holds. Furthermore, it is obvious if $trace_g \varphi = const$, then φ^{TT} is also a Killing tensor. Theorem 2 is proven.

Next, let us consider equation (1) of a Codazzi tensor φ. From (1) we obtain

$$\delta\, \varphi = -\, d\big(trace_g \varphi\big).$$

In this case equation (10) has the form

$$\delta \overset{\circ}{\varphi} = -\, \frac{n-1}{n}\, d\big(trace_g \varphi\big).$$

In turn, the integral formula (11) can be rewritten in the form

$$\langle S\theta, S\theta\rangle = -\frac{n-1}{n} \int_M L_\xi \big(trace_g \varphi\big) dvol_g.$$

If we assume that $\int_M L_\xi \big(trace_g \varphi\big) dvol_g \geq 0$, then from the last formula we obtain that $S\theta = 0$ and $\int_M L_\xi \big(trace_g \varphi\big) dvol_g = 0$. In this case, we have $\overset{\circ}{\varphi} = \varphi^{TT}$ and hence (3) holds. Furthermore, it is obvious if $trace_g \varphi = const$, then φ^{TT} is also a Codazzi tensor. Theorem 1 is proven.

In conclusion, we consider the Ricci tensor Ric. As can be seen from the well-known second Bianchi identity, one has

$$\delta\, Ric = -\frac{1}{2}\, ds.$$

where $s$is the scalar curvature, defined as $s = trace_g Ric$. In this case (8) can be rewritten in the form

$$\overset{\circ}{Ric} = \ S\theta + \varphi^{TT}$$

and hence

$$\langle S\theta, S\theta\rangle = -\frac{n-2}{n} \int_M \big(L_\xi\ s\ \big) dvol_g,$$

respectively. Therefore, if we assume that $n \geq 3$ and $\int_M \big(L_\xi\ s\ \big) dvol_g \geq 0$, then from the last formula we obtain $S\theta = 0$

and $\int_M L_\xi \, s \, dvol_g = 0$. Therefore, ξ is a conformal Killing vector field and $Ric = \frac{1}{n} s \, g + \varphi^{TT}$, where s must be constant.

References

1. *Besse, A. L.:* Einstein manifolds. Springer-Verlag, Berlin & Heidelberg (2008).

2. *Petreson, P.:* Riemannian Geometry. Springer International Publishing AG (2016).

3. *Gicquaud, R., Ngo, Q. A.:* A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small *TT*-tensor. Class. Quant. Grav., **31**:19, 195014 (2014).

4. *Bourguignon, J. P., Ebin, D. G., Marsden, J. E.:* Sur le noyau des opérateurs pseudo-differentiels á symbole surjectif et non injectif. Comptes rendus hebdomadaires des séances de l'Académie des sciences. Sér. A et B, Sciences mathématiques et Sciences physiques, 282, 867—870 (1976).

5. *Garattini, R.:* Self sustained tranversable wormholes? Class. Quant. Grav., **22**:6, 2673—2682 (2005).

6. *York, J. W.:* Covariant decompositions of symmetric tensors in the theory of gravitation. Ann. Inst. H. Poincaré Sect. A (N. S.), **21**:4, 319—332 (1974).

7. *Carlotto, A.:* The general relativistic constraint equations. Living Reviews in Relativity, **24**:2, 1—170 (2021).

8. *Stepanov, S. E., Tsyganok, I. I.:* Pointwise orthogonal splitting of the space of TT-tensors. DGMF, 54 (2), 45—53 (2023).

9. *Simons, J.:* Minimal varieties in Riemannian manifolds. Ann. Math., **88**:1, 62—105 (1968).

10. *Penrose, R., Walker, M.:* On quadratic first integrals of the geodesic equations for type {22} spacetimes. Commun. Math. Phys., 18, 265—274 (1970).

11. *Stepanov, S. E., Smol'nikova, M. V.:* Affine differential geometry of Killing tensors. Russian Math. (Izvestia Vuzov), **48**:11, 74—78 (2004).

12. *Stepanov, S. E., Tsyganok, I., Khripunova, M.:* The Killing tensor on an n-dimensional manifold with $SL(n, R)$-structure. Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica, **55**:1, 121—131 (2016).

13. *Branson, T.:* Stein-Weiss operators and ellipticity. J. Funct. Anal., 151, 334—383 (1997).

For citation: Stepanov, S. E., Tsyganok, I. I.: York decompositions for the Codazzi, Killing and Ricci tensors. DGMF, 56, 119—127 (2025). https://doi.org/10.5922/0321-4796-2025-56-10.

УДК 514.764

С. Е. Степанов[1], *И. И. Цыганок*[2]
[1, 2] *Финансовый университет при Правительстве РФ, Москва, Россия*
[1] s.e.stepanov@mail.ru, [2] i.i.tsyganok@mail.ru
doi: 10.5922/0321-4796-2025-56-10

Разложения Йорка для тензоров Кодацци, Киллинга и Риччи

Поступила в редакцию 15.12.2024 г.

Разложение Йорка пространства симметричных 2-тензоров возникло в теоретической физике и нашло применение в римановой геометрии, как это показано в известной монографии Бессе о многообразиях Эйнштейна. В этой статье мы выводим разложения Йорка для тензоров Кодацци, Киллинга и Риччи на замкнутом римановом многообразии. В частности, мы выводим разложения Йорка для 2-тензоров Кодацци, Киллинга и Кодацци с постоянными следами.

Ключевые слова: замкнутое риманово многообразие, разложение Йорка, тензор Кодацци, тензор Киллинга

Список литературы

1. *Besse A. L.* Einstein manifolds. Berlin ; Heidelberg, 2008.
2. *Petreson P.* Riemannian Geometry. Springer International Publishing AG, 2016.
3. *Gicquaud R., Ngo Q. A.* A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor // Class. Quant. Grav. 2014. Vol. 31, № 19. P. Art № 195014.

4. *Bourguignon, J.P., Ebin D. G., Marsden J.E.* Sur le noyau des opérateurs pseudo-differentiels á symbole surjectif et non injectif // Comptes rendus hebdomadaires des séances de l'Académie des sciences. Sér. A et B, Sciences mathématiques et Sciences physiques. 1976. Vol. 282. P. 867—870.

5. *Garattini R.* Self sustained tranversable wormholes? // Class. Quant. Grav. 2005. Vol. 22, №6. P. 2673—2682.

6. *York J.W.* Covariant decompositions of symmetric tensors in the theory of gravitation // Ann. Inst. H. Poincaré Sect. A (N.S.). 1974. Vol. 21, №4. P. 319—332.

7. *Carlotto A.* The general relativistic constraint equations // Living Reviews in Relativity. 2021. Vol. 24, №2. P. 1—170.

8. *Степанов С.Е., Цыганок И.И.* Поточечное расщепление пространства TT-тензоров // ДГМФ. 2023. №54 (2). С. 45—53.

9. *Simons J.* Minimal varieties in Riemannian manifolds // Ann. Math. 1968. Vol. 88, №1. P. 62—105.

10. *Penrose R., Walker M.* On quadratic first integrals of the geodesic equations for type {22} spacetimes // Commun. Math. Phys. 1970. Vol. 18. P. 265—274.

11. *Степанов С.Е., Смольникова М.В.* Аффинная дифференциальная геометрия тензоров Киллинга // Изв. вузов. Математика. 2004. №11. С. 82—86.

12. *Stepanov S.E., Tsyganok I., Khripunova M.* The Killing tensor on an n-dimensional manifold with $SL(n, R)$-structure // Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica. 2016. Vol. 55, №1. P. 121—131.

13. *Branson T.* Stein-Weiss operators and ellipticity // Journal of Functional Analysis. 1997. Vol. 151. P. 334—383.

Для цитирования: *Степанов С.Е., Цыганок И.И.* Разложения Йорка для тензоров Кодацци, Киллинга и Риччи // ДГМФ. 2025. №56. С. 119—127. https://doi.org/10.5922/0321-4796-2025-56-10.