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I. S. Vereshchagina, S. D. Vereshchagin

ON THE PROBLEM OF EVALUATING THE ACCURACY OF DIAGNOSTICS OF WAVE DISTURBANCES CARRIED OUT USING THE TECHNIQUE OF PROJECTION OPERATORS

In this note we study the problem of a function reconstruction in a context of a Laplace method application. We use a unitary space of splines with a double dimension of one, that approximate the set of points, representing the results of observation. The conventional scalar product allows to project the approximation onto the subspace of observations. The use of the same scalar product yields the norm that we use to estimate error deviations within the model under consideration. Its minimum defines both a function reconstruction and its error, which also include the measurements errors. The results we apply to the problems of reconstruction of initial or boundary conditions for 1D wave equation, that imply the procedure of directed waves division.

Изучается проблема восстановления функции в контексте применения метода Лапласа. Мы используем унитарное пространство сплайнов с удвоением количества, которое аппроксимирует множество точек, представляющих результаты наблюдения. Обычное скалярное произведение позволяет проецировать приближение на подпространство наблюдений. Использование того же скалярного произведения дает норму, которую мы используем для оценки отклонений ошибок в рассматриваемой модели. Его минимум определяет как восстановление функции, так и ее ошибку, которая также включает ошибки измерений. Полученные результаты применимы к задачам восстановления начальных или граничных условий для одномерного волнового уравнения, предполагающих процедуру разделения направленных волн.

Keywords: atmosphere, diagnostics of disturbances, projection operators, math modeling.

Ключевые слова: атмосфера, диагностика возмущений, операторы проектирования, математическое моделирование.

Introduction

The problem we touch relates to sampling theory and interpolation, with some focus on the Shannon — Niquist — Kotelnikov theorem [1; 2]. We do restrict ourselves by practical aims, having in mind estimations of intermediate ordinates between observed values of a function that represent wave phenomena [3; 4]. The second paper [4] use the Fourier basis and state, that for unambiguous restoration of a continuous signal from its samples need to double the sampling rate maximum frequency in the signal spectrum. The procedure we propose consumes the Laplace — Legendre ideas about minimization of a functional space distance between the continuous function representation by 2n-dim-splines and n-dim splines that mimic observations. The procedure implies a definition of projection the 2n-dim space onto the n-dim space that fix intermediate ordinates of the wave function under consideration. The minimization gives simple system of equations, that contain 2nxn matrix A, hence the problem is classical ill-posed one. Its simplest regularization is given by $A^t A$, that determines quasi-solution of the problem [5]. Its minimum also defines both a function reconstruction and its error, which also includes the measurements errors. The results we apply to the problems of reconstruction of initial or boundary conditions for 1D wave equation, that imply the procedure of directed waves division within the dynamic projecting method [6].

Dynamic projection operator method

The main idea of the method of projection operators is to divide the solution space into subspaces of solutions corresponding to various branches of the dispersion relation [7]. To do this, it is necessary to present the original problem in matrix form

$$\psi_t = L\psi$$

with the evolution operator L and a state ψ of a system $\psi = \begin{pmatrix} V \\ p \end{pmatrix}$.

The Fourier transformation $V(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}(k) e^{ikx}\, dk$ may be written as the matrix substitution $\psi = F\tilde{\psi}$ and describes transition to k-representation of the evolution operator $\tilde{L}$:

$$\tilde{\Psi}_t = F^{-1} L F \tilde{\psi} = \tilde{L}\tilde{\psi}.$$

The matrix eigenvalue problem $\tilde{L}\phi = \lambda\phi$ introduces subspaces, which we would represent by the matrix of solutions Ψ, so that $\tilde{L}\Psi = \Psi\Lambda$, where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2\}$ — diagonal matrix. If $\lambda_1 \neq \lambda_2$ (eigenvalues) the inverse matrix exists and $\tilde{L} = \Psi\Lambda\Psi^{-1}$.

Spectral decomposition of the matrix L

$$L_{ij} = \psi_{ik}\Lambda_{kl}\psi_{lj}^{-1} = \psi_{ik}\lambda_k\psi_{kj}^{-1} = \sum_k \lambda_k \psi_{ik}\psi_{kj}^{-1} = \sum_k \lambda_k \left(P_k\right)_{ij}.$$

The projection operator can be also found using the relations and properties of the projection operators $\tilde{P}_i \; \psi = \psi_i$, ψ_i — eigenvectors of the evolution matrix $\tilde{L}$. Properties of the projection operator:

$$\tilde{P}_i * \tilde{P}_j = 0, \quad \tilde{P}_i^{\,2} = \tilde{P}_i, \quad \sum_i \tilde{P}_i = 1.$$

The spectral decomposition of the matrix [6] allows to find the projection operator through the direct product

$$\tilde{P}_i = \psi_i \otimes \Psi_i^{-1},$$

ψ_i — i-th column; ψ_i^{-1} — i-th row of inverse Fourier transform of Ψ.

Diagnostics of wave disturbances

One-dimensional adiabatic acoustic wave propagation for the ideal gas can be represented as solution of the system

$$\begin{cases} V_t - cp_x = 0, \\ p_t - cV_x = 0. \end{cases}$$

The initial conditions

$$p(x,0) = \varphi_1(x), \; V(x,0) = \varphi_2(x),$$

define the Cauchy problem.

The evolutionary equation has the form $\psi_t = L\psi$, where

$$\psi = \begin{pmatrix} v \\ p \end{pmatrix}, \; L = \mathrm{cD}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \mathrm{D} = \frac{\partial}{\partial x}.$$

Projection operators of one-dimensional adiabatic acoustic wave propagation for the ideal gas looks as $P_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$.

The general solution is determined by the relation $(P_+ + P_-)\psi = \psi$.

Acting as a design operator, we select a unidirectional wave

$$P_+\psi = \begin{pmatrix} \psi_+ \\ \psi_+ \end{pmatrix}, \; P_-\psi = \begin{pmatrix} \psi_- \\ -\Psi_- \end{pmatrix}.$$

For a one-dimensional adiabatic acoustic problem, this allows us to determine the coupling equations and evolution equations for unidirectional waves

$$\psi_+ = \frac{1}{2}(p+v); \; \psi_- = \frac{1}{2}(p-v);$$

$$(\Psi_+)_t + c(\psi_+) = 0,$$

$$(\Psi_-)_t - c(\psi_-) = 0.$$

Evaluation the accuracy of diagnostics of wave disturbances

Let we have some set of datapoints $y_i = y(x = x_i)$ defined in the points $x = x_i \in [0,1], i = 0..n$. They can, for example, be result of application of some projection operators to data in some diagnostic problem, but in fact their origin does not really matter. Its spline representation is constructed as

$$\phi = \sum_{i=1}^{n} \eta_i \tilde{s}_i(x) \in S^n, \quad \text{where } \tilde{s}_i(x) = \begin{cases} 1, x \in \left[\frac{i}{n}, \frac{i+1}{n}\right]. \\ 0, other\ wise \end{cases}$$

The scalar product in S is defined as

$$(\phi, \phi_1) = \int_0^1 \phi(x)\phi_1(x)\,dx .$$

Next, we search the solution of the problem in the space S^{2n} via splines

$$\psi = \sum_{i=1}^{2n} \xi_i s_i(x) \in S^{2n},$$

where its projection to the S^n is defined by the relation

$$\xi = \frac{\xi_i + \xi_{i+1}}{2},$$

hence

$$\psi_+ = \sum_{i=1}^{2n} \frac{\xi_i + \xi_{i+1}}{2} \tilde{s}_i(x) \in S^n,$$

whence its components are calculated via the scalar product

$$\left(\tilde{s}_l, \psi_+\right) = \psi_{l+}.$$

Let's first apply proposed method to the simplest possible case where we have two points with values A and B separated by the distance h on the OX axis (i. e. their x — coordinates are x_0 and $x_0 + h$).

We will use piecewise-constant approximation for our function. To do it we can introduce new point situated at the midpoint between our initial points with value

$$C = \frac{A+B}{2}.$$

In a case we have some background information about our function method, it can be modified by using another value for C but here we do not have it and half-point should work good enough.

Now we will try to construct another approximation of our function. Simplest is two-point piecewise-constant, where function has one value, let's call it X, for the first half of our interval and the second value, Y, on the second half.

Now we will try to find X and Y in such a way as to minimize a norm

$$E = \int_{x_0}^{x_0+\frac{h}{3}} (A-X)^2\,dx + \int_{x_0+\frac{h}{3}}^{x_0+\frac{h}{2}} (C-X)^2\,dx + \int_{x_0+\frac{h}{2}}^{x_0+\frac{2h}{3}} (C-Y)^2\,dx + \int_{x_0+\frac{2h}{3}}^{x_0+h} (B-Y)^2\,dx.$$

Since they are independent we can separately solve this for X and Y. We'll get

$$X = \frac{5A+B}{6}, \qquad Y = \frac{A+5B}{6},$$

then minimal possible value of norm is

$$N_{min} = \frac{(A-B)^2 h}{18}.$$

Now if we have a set of datapoints we can apply this formula at each interval (x_i, x_{i+1}) independently. That means that, assuming all intervals have the same length h, that global error over all datapoints take the form

$$E = \sum_{i=0}^{N-1} \frac{h}{18}(y_i - y_{i+1})^2 = \frac{h}{18}\sum_{i=0}^{N-1}(y_i - y_{i+1})^2 = \frac{L}{18N}\sum_{i=0}^{N-1}(y_i - y_{i+1})^2, \qquad (1)$$

where $L = x_N - x_0$ — distance between first and last points of our set.

If we assume that the modulus of the derivative is always less than some constant Z then when h is small

$$E = \frac{L}{18N}\sum_{i=0}^{N-1}(y_i - y_{i+1})^2 \le \frac{L}{18N}\sum_{i=0}^{N-1}(Zh)^2 = \frac{L}{18N}(N-1)Z^2h^2 \approx \frac{L^3}{18N^2}Z^2,$$

so at least this method pass sanity check — error goes down with increase of number of data points.

Let's look how does it work on simplest example — string equation. Equation of string have the form

$$\frac{\partial^2 U(x,t)}{\partial t^2} = c^2 \frac{\partial^2 U(x,t)}{\partial x^2}.$$

We can rewrite this equation as a system of two first order ordinary differential equations over $U(x,t)$ and $V(x,t) = \frac{\partial U(x,t)}{\partial t}$. In this case it is trivial to obtain projection operators (for procedure look for example [6])

$$P_{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

The general solution in this case take form of sum of left-and right-running waves:

$$F = \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} F_+ \\ F_+ \end{pmatrix} + \begin{pmatrix} F_- \\ -F_- \end{pmatrix},$$

where F_+ and F_- are arbitrary functions. These projection operators does indeed separate our functions into two waves with different directions. While we act in the space of symbolical functions this result is exact.

Now let us choose for example $F_+ = F_0$ and fix the point of «measurement» $x = x_0$. Then we can generate a series of N datapoints taking values of one of our waves in the points $t = i \cdot h$, where $i = 1..N$ is the number of point, $h = \frac{L}{N}$ is the step between points and L is the length of the time interval we are interested in.

$$F_+[n] = \begin{pmatrix} F_0(kh) \\ F_0(kh) \end{pmatrix}.$$

Then we can try to restore continuous function by pulling splines over datapoints and then we can calculate of error of this restoration according to formula (1).

If we choose $F_0 = exp\left(-\left(x - c \cdot t - 5\right)^2\right)$ and $x_0 = 5$ then depending on the number of points we take our error is presented on the graphics (Fig. 1) (horizontal axis is the common logarithm of number of points):

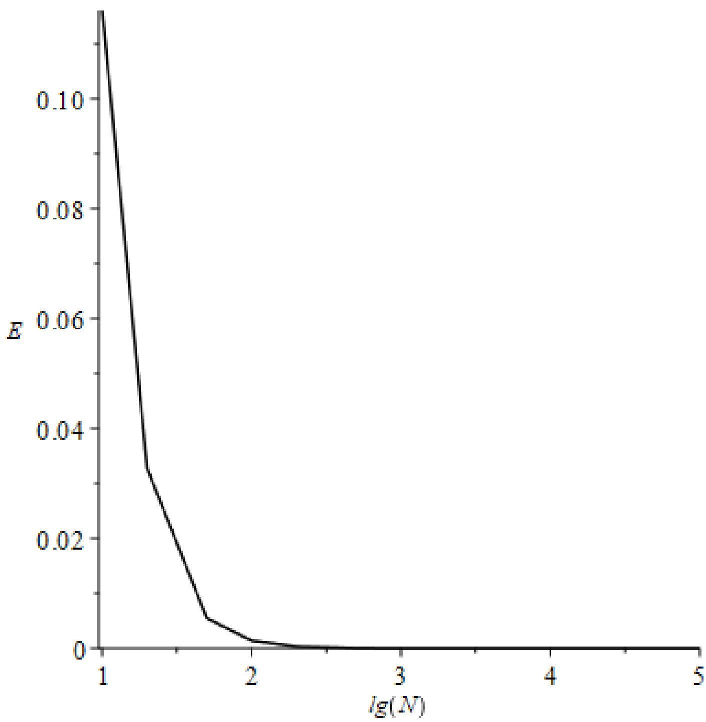

Fig. 1. Error as function of number of points for Gaussian

If our function is not as smooth as Gaussian then error, predictably, is a log bigger. For example if we choose fast oscillating function $F_0 = exp\left(-\left(x - c \cdot t - 5\right)^2\right) \cdot cos(30t)$ we 'll get (Fig. 2).

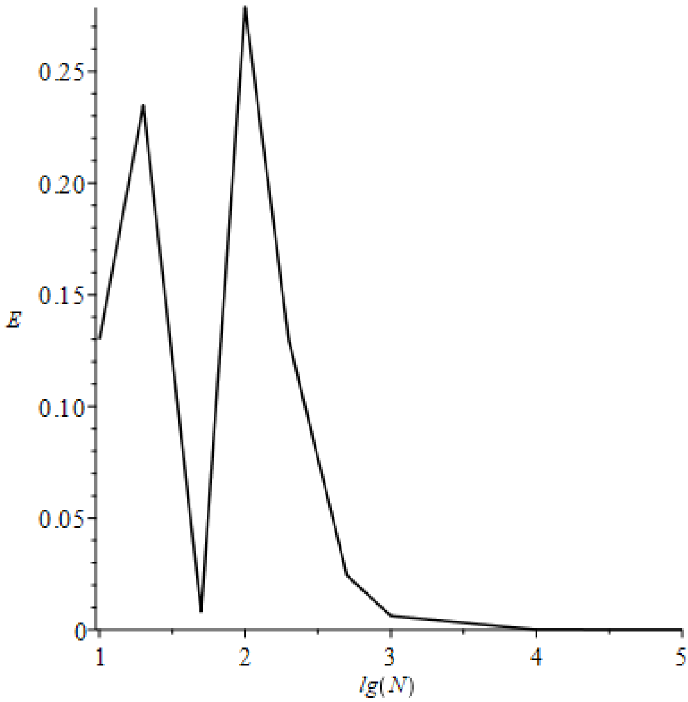

Fig. 2. Error as function of number of points for fast oscillating function

Of course, because of Shannon — Niquist — Kotelnikov theorem we can't really use the part of this graphics which is to the left of $lg(N)=2$. As can be seen the error does indeed goes to zero when the number of points grow. If we can choose N, for example when we are deciding on the parameters of the numerical model, this method, following the general idea of the Runge's rule, can be one of deciding factor in choosing model's number of points. If it is applied to the experimental data, where number of points is generally fixed, it can be used to determine an error of reconstruction of continuous function.

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The authors

Dr Irina S. Vereshchagina, Associate Professor, Immanuel Kant Baltic Federal University, Russia.

E-mail: ver_is@mail.ru

Dr Sergey D. Vereshchagin, Associate Professor, Immanuel Kant Baltic Federal University, Russia.

E-mail: sergey.ver@gmail.com

Об авторах

Ирина Сергеевна Верещагина — канд. физ.-мат. наук, доц., Балтийский федеральный университет им. И. Канта, Россия.

E-mail: ver_is@mail.ru

Сергей Дмитриевич Верещагин — канд. физ.-мат. наук, доц., Балтийский федеральный университет им. И. Канта, Россия.

E-mail: sergey.ver@gmail.com