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GENERATING FUNCTION AND ITS APPLICATION FOR GENERALIZATION OF LEGENDRE POLYNOMIALS

Проводится обобщение полиномов Лежандра с помощью производящей функции. Найден явный вид искомых функций. Рассмотрены некоторые частные случаи. Особое внимание уделено случаю, когда параметры, определяющие изучаемые функции, являются различными, симметричными относительно нуля действительными числами. Изучены некоторые свойства этих функций. Основываясь на результатах численных экспериментов, выдвинута гипотеза о корнях исследуемых функций.

In the present paper the generalization of Legendre polynomials with the help of generating function is studied. The explicit form of considered functions is found. Some special cases are considered. Particular attention is paid to the case when the parameters defining the studied functions are different, symmetric about zero real numbers. Some properties of constructed functions are obtained. Based on the results of numerical experiment a hypothesis about zeroes of these functions is stated.

Ключевые слова: полиномы Лежандра, производящая функция, ортогональные системы.

Key words: Legendre polynomials, generating function, orthogonal systems.

Introduction

Generating function is one of the classical ways for construction of orthogonal systems. Application of generating functions for construction of orthogonal systems of algebraic polynomials is described in detail in the [1]. In the case of rational functions this problem is more complicated. In 1964 M. M. Dzhrbashyan and A. A. Kitbalyan applied generating function for construction of systems of rational functions which generalized Chebyshev polynomials of the first and the second type [2]. It should be mentioned that in the work [3] the construction of system of rational functions which generalized Jacobi polynomials with respect to the weight $\sqrt{(1-x)/(1+x)}$ was described.

In the present paper the generalization of Legendre polynomials with the help of generating function is considered.

It's well known that the functions

$$F(x,z)=\frac{1}{\sqrt{1-2xz+z^2}},\ x\in(-1,1),\ F(x,0)=1,$$

is a generating function for algebraic Legendre polynomials, i. e.

$$F(x,z)=\sum_{n=0}^{\infty}\frac{P_n(x)}{n!}z^n\ ,\ |z|<1\ .$$

In other words, the Legendre polynomials are the Taylor coefficients of the function $F(x,z)$,

$$P_n(x)=\frac{1}{2\pi i}\int_{|z|=\rho}F(x,z)\frac{dz}{z^{n+1}}\ ,\ 0<\rho<1\ .$$

1. Main result

For the generalization of Legendre polynomials on rational case we consider the following rational functions. Let the sequence $\{\alpha_k\}_{k=0}^{\infty}$ of complex numbers be such that $\alpha_0=0$, $|\alpha_k|<1$, $k\in\mathbb{N}_0$. Using this numbers we define the functions:

$$\varphi_0(z)\equiv 1\ ,\ \varphi_n(z)=\frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha_n}z}\prod_{k=0}^{n-1}\frac{z-\alpha_k}{1-\overline{\alpha_k}z}\ ,\ n\in\mathbb{N}_0.$$

Lemma 1 [1]. *The system* $\{\varphi_n(z)\}_{n=0}^{\infty}$ *is a orthogonal on the unit circle* $|z|=1$.

Note, that if $|z|=1$ then

$$\overline{\varphi_n(z)}=\frac{\sqrt{1-|\alpha_n|^2}}{1-\alpha_n\overline{z}}\prod_{k=0}^{n-1}\frac{\overline{z}-\overline{\alpha_k}}{1-\alpha_k\overline{z}}=\frac{\sqrt{1-|\alpha_n|^2}}{z-\alpha_n}\prod_{k=1}^{n-1}\frac{1-\overline{\alpha_k}z}{z-\alpha_k}\ . \tag{1}$$

Here and later the function $\overline{\varphi_n(z)}$ is defined by formula (1) also in the case $|z|\neq 1$.

The functions, which generalize the Legendre polynomials, we define as follows

$$L_n(x)=\frac{1}{2\pi i}\int_{\Gamma}F(x,z)\overline{\varphi_n(z)}\frac{dz}{z}\ , \tag{2}$$

where Γ is a circle $|z|=\rho$, $0<\rho<1$, such that $|\alpha_k|<\rho$, $k=1,2,...,n$.

Remark. *If* $\alpha_k=0$, $k=1,2,...,n$, *then* $\overline{\varphi_n(z)}=\frac{1}{z^n}$, *and* $L_n(x)=\frac{1}{2\pi i}\int_{|z|=\rho}F(x,z)\frac{dz}{z^{n+1}}$, *i. e. in this case* $L_n(x)$ *is a Legendre polynomials, orthogonal on the segment* $[-1,1]$ *with respect to the weight* 1.

Theorem 1. *If* $\alpha_k\neq 0$, $k=1,2,...,n$, *and* $\alpha_k\neq\alpha_j$, $k\neq j$, $k,j=1,2,...,n$, *then the following formula holds*

$$L_n(x)=\sqrt{1-|\alpha_n|^2}\left(\frac{(-1)^n}{\prod_{k=1}^{n}\alpha_k}+\sum_{k=1}^{n-1}\frac{1-|\alpha_k|^2}{\alpha_k(\alpha_k-\alpha_n)}\frac{1}{\sqrt{1-2x\alpha_k+\alpha_k^2}}\prod_{\substack{j=1\\ j\neq k}}^{n-1}\frac{1-\overline{\alpha_j}\alpha_k}{\alpha_k-\alpha_j}+\right.$$
$$\left.+\frac{1}{\alpha_n\sqrt{1-2x\alpha_n+\alpha_n^2}}\prod_{j=1}^{n-1}\frac{1-\overline{\alpha_j}\alpha_n}{\alpha_n-\alpha_j}\right) \tag{3}$$

Proof. Using formula (1) we obtain

$$L_n(x)=\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\sqrt{1-2xz+z^2}}\frac{\sqrt{1-|\alpha_n|^2}}{z-\alpha_n}\prod_{k=1}^{n-1}\frac{1-\overline{\alpha_k}z}{z-\alpha_k}\frac{dz}{z}.$$

Then we apply the substitution

$$\sqrt{1-2xz+z^2}=1-zt \text{ or } z=\frac{2(t-x)}{t^2-1}. \tag{4}$$

In this case

$$dz=-2\frac{t^2-2tx+1}{(t^2-1)^2}dt,\ \sqrt{1-2xz+z^2}=-\frac{t^2-2tx+1}{t^2-1},$$

$$\frac{1-\overline{\alpha_k}z}{z-\alpha_k}=\frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)},\ z-\alpha_n=\frac{\alpha_n(1-t^2)+2(t-x)}{t^2-1}.$$

Therefore

$$L_n(x)=\frac{\sqrt{1-|\alpha_n|^2}}{2\pi i}\int_C\frac{t^2-1}{t-x}\frac{1}{\alpha_n(1-t^2)+2(t-x)}\prod_{k=1}^{n-1}\frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)}dt,$$

where C is an image of Γ when mapping $t(z)$.

Under the conditions of the theorem the integrand

$$F(t)=\frac{t^2-1}{t-x}\frac{1}{\alpha_n(1-t^2)+2(t-x)}\prod_{k=1}^{n-1}\frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)}$$

has simple poles at the point $t=x$ and at the roots of the equations

$$\alpha_k(1-t^2)+2(t-x),\ k=1,2,...,n,$$

i. e.

$$t_{k,1}=\frac{1-\sqrt{1-2x\alpha_k+\alpha_k^2}}{\alpha_k},\ t_{k,2}=\frac{1+\sqrt{1-2x\alpha_k+\alpha_k^2}}{\alpha_k},\ k=1,2,...,n.$$

It is not difficult to show that the points $t_{k,1}$, $k=1,2,...,n$, are inside the curve C, and the points $t_{k,2}$, $k=1,2,...,n$, are outside this curve. To prove it one needs to find the corresponding points $z_{k,j}=z(t_{k,j})$, $k=1,2,...,n$, $j=1,2$ using (4) and verify that $z_{k,1}z_{k,2}=1$, $|z_{k,1}|<1$, $k=1,2,...,n$.

By the Cauchy's residue theorem we get

$$L_n(x)=\sqrt{1-|\alpha_n|^2}\left(\operatorname{Res}(F,x)+\sum_{k=1}^{n}\operatorname{Res}(F,t_{k,1})\right). \tag{5}$$

It is easy to see that

$$\operatorname{Res}(F,x)=\frac{(-1)^n}{\prod_{k=1}^{n}\alpha_k}. \tag{6}$$

Now we calculate residues at the points $t_{k,1}$, $k=1,2,...,n-1$. We have

$$\operatorname{Res}(F,t_{k,1})=\left.\frac{t^2-1}{t-x}\right|_{t=t_{k,1}}\left.\frac{1}{\alpha_n(1-t^2)+2(t-x)}\right|_{t=t_{k,1}}\times\prod_{\substack{j=1\\ j\neq k}}^{n-1}\left.\frac{t^2-1-2\overline{\alpha_j}(t-x)}{\alpha_j(1-t^2)+2(t-x)}\right|_{t=t_{k,1}}\left.\frac{t^2-1-2\overline{\alpha_k}(t-x)}{-\alpha_k(t-t_{k,2})}\right|_{t=t_{k,1}}.$$

Then we find $\left.\frac{t^2-1}{t-x}\right|_{t=t_{k,1}}=\frac{2}{\alpha_k}$,

$$\left.\frac{1}{\alpha_n(1-t^2)+2(t-x)}\right|_{t=t_{k,1}}=\left.\frac{1}{\left(\alpha_n\frac{1-t^2}{t-x}+2\right)(t-x)}\right|_{t=t_{k,1}}=$$

$$=\frac{1}{\left(-\alpha_n\frac{2}{\alpha_k}+2\right)\left(\frac{1-\sqrt{1-2x\alpha_k+\alpha_k^2}}{\alpha_k}-x\right)}=\frac{\alpha_k^2}{2(\alpha_k-\alpha_n)\left(1-\alpha_k x-\sqrt{1-2x\alpha_k+\alpha_k^2}\right)},$$

$$\left.\frac{t^2-1-2\overline{\alpha_j}(t-x)}{\alpha_j(1-t^2)+2(t-x)}\right|_{t=t_{k,1}}=\left.\frac{\frac{t^2-1}{t-x}-2\overline{\alpha_j}}{\alpha_j\frac{1-t^2}{t-x}+2}\right|_{t=t_{k,1}}=\frac{\frac{2}{\alpha_k}-2\overline{\alpha_j}}{-\alpha_j\frac{2}{\alpha_k}+2}=\frac{1-\overline{\alpha_j}\alpha_k}{\alpha_k-\alpha_j},$$

$$\left.\frac{t^2-1-2\overline{\alpha_k}(t-x)}{-\alpha_k(t-t_{k,2})}\right|_{t=t_{k,1}}=\left.\frac{\left(\frac{t^2-1}{t-x}-2\overline{\alpha_k}\right)(t-x)}{-\alpha_k(t-t_{k,2})}\right|_{t=t_{k,1}}=$$

$$=\frac{\left(\frac{2}{\alpha_k}-2\overline{\alpha_k}\right)\left(\frac{1-\sqrt{1-2x\alpha_k+\alpha_k^2}}{\alpha_k}-x\right)}{-\alpha_k\left(\frac{1-\sqrt{1-2x\alpha_k+\alpha_k^2}}{\alpha_k}-\frac{1+\sqrt{1-2x\alpha_k+\alpha_k^2}}{\alpha_k}\right)}=\frac{\left(1-\alpha_k x-\sqrt{1-2x\alpha_k+\alpha_k^2}\right)(1-|\alpha_k|^2)}{\alpha_k^2\sqrt{1-2x\alpha_k+\alpha_k^2}}.$$

Since that,

$$\operatorname{Res}(F,t_{k,1})=\frac{1-|\alpha_k|^2}{\alpha_k(\alpha_k-\alpha_n)}\frac{1}{\sqrt{1-2x\alpha_k+\alpha_k^2}}\prod_{\substack{j=1\\j\neq k}}^{n-1}\frac{1-\overline{\alpha_j}\alpha_k}{\alpha_k-\alpha_j}. \tag{7}$$

And the residue at the point $t_{n,1}$

$$\operatorname{Res}(F,t_{n,1})=\left.\frac{t^2-1}{t-x}\right|_{t=t_{n,1}}\left.\frac{1}{-\alpha_n(t-t_{n,2})}\right|_{t=t_{n,1}}\prod_{j=1}^{n-1}\left.\frac{t^2-1-2\overline{\alpha_j}(t-x)}{\alpha_j(1-t^2)+2(t-x)}\right|_{t=t_{n,1}}=$$
$$=\frac{1}{2\sqrt{1-2x\alpha_n+\alpha_n^2}}\prod_{j=1}^{n-1}\frac{1-\overline{\alpha_j}\alpha_n}{\alpha_n-\alpha_j}. \tag{8}$$

To complete the proof one must put (6), (7) and (8) into (5). □

Corollary 1. *By the formula* (3) *the function* $L_n(x)$ *can be written in the form*

$$L_n(x)=\sum_{k=0}^{n}\frac{c_k}{\sqrt{1-2x\alpha_k+\alpha_k^2}},$$

where c_k, $k=1,2,\ldots,n$, *are some complex numbers. Its singular points are as follows*

$$x_k=\frac{1+\alpha_k^2}{2\alpha_k},\ k=1,2,\ldots,n.$$

2. Some important cases

1) Obviously, $L_0(x) \equiv 1$.

2) Let $\alpha_0 = 0$, $\alpha_1 = \alpha$, $|\alpha| < 1$. Then the second summand in the right side of (3) vanishes, and the product in the third summand is equal to 1. Since that,

$$L_1(x) = \frac{\sqrt{1-|\alpha|^2}}{\alpha}\left(-1+\frac{1}{\sqrt{1-2x\alpha+\alpha^2}}\right).$$

Note that

$$\lim_{\alpha\to 0} L_1(x) = \lim_{\alpha\to 0}\frac{1-\sqrt{1-2x\alpha+\alpha^2}}{\alpha} = \lim_{\alpha\to 0}\frac{2x\alpha-\alpha^2}{\alpha\left(1+\sqrt{1-2x\alpha+\alpha^2}\right)} = x .$$

Besides, the function $L_1(x)$ has the only zero at the point

$$x = \frac{\alpha}{2} .$$

3) Let the sequence $\{\alpha_k\}_{k=0}^{n}$ be as follows: $\alpha_0 = 0$, $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ (distinct points), $\alpha_n = 0$. In this case

$$\varphi_n(z) = \prod_{k=0}^{n-1}\frac{z-\alpha_k}{1-\overline{\alpha_k}z} , \ \overline{\varphi_n(z)} = \frac{1}{z}\prod_{k=1}^{n-1}\frac{1-\overline{\alpha_k}z}{z-\alpha_k}$$

and in a similar way

$$L_n(x) = \frac{1}{4\pi i}\int_C \frac{t^2-1}{(t-x)^2}\prod_{k=1}^{n-1}\frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)}dt$$

or

$$L_n(x) = \frac{1}{2}\left(\mathrm{Res}(F,x)+\sum_{k=1}^{\mathrm{n-1}}\mathrm{Res}(F,t_{k,1})\right),$$

where

$$F(t) = \frac{t^2-1}{(t-x)^2}\prod_{k=1}^{n-1}\frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)},$$

$t_{k,1}$ is the root of the equation $\alpha_k(1-t^2)+2(t-x)=0$ which lies inside the curve *C*.

Now we find residues. The point $t = x$ is a pole of the second order. Thus

$$\mathrm{Res(F,x)} = \left((t^2-1)\prod_{k=1}^{n-1}\frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)}\right)'_{t=x} =$$

$$= 2x\prod_{k=1}^{n-1}\frac{x^2-1}{\alpha_k(1-x^2)} + (x^2-1)\sum_{j=1}^{n-1}\prod_{\substack{k=1\\k\neq j}}^{n-1}\frac{x^2-1}{\alpha_k(1-x^2)}\left(\frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)}\right)'_{t=x} =$$

$$= \frac{(-1)^{n-1}2x}{\prod_{k=1}^{n-1}\alpha_k} + (x^2-1)\sum_{j=1}^{n-1}\frac{(-1)^n}{\prod_{\substack{k=1\\k\neq j}}^{n-1}\alpha_k}\frac{2(1-|\alpha_j|^2)}{\alpha_j^2(1-x^2)} = \frac{(-1)^{n-1}2}{\prod_{k=1}^{n-1}\alpha_k}\left(x+\sum_{j=1}^{n-1}\frac{1-|\alpha_j|^2}{\alpha_j}\right).$$

The points $t_{k,1}$, $k = 1,2,...,n-1$, are the simple poles. We have

$$\mathrm{Res}(F, t_{k,1}) = \left(\frac{t^2-1}{(t-x)^2} \frac{t^2-1-2\overline{\alpha_k}(t-x)}{-\alpha_k(t-t_{k,2})} \prod_{\substack{j=1\\ j\neq k}}^{n-1} \frac{t^2-1-2\overline{\alpha_k}(t-x)}{\alpha_k(1-t^2)+2(t-x)} \right)_{t=t_{k,1}} =$$

$$= \frac{2}{1-\alpha_k x - \sqrt{1-2x\alpha_k+\alpha_k^2}} \frac{\left(1-\alpha_k x - \sqrt{1-2x\alpha_k+\alpha_k^2}\right)(1-|\alpha_k|^2)}{\alpha_k^2\sqrt{1-2x\alpha_k+\alpha_k^2}} \prod_{\substack{j=1\\ j\neq k}}^{n-1} \frac{1-\overline{\alpha_j}\alpha_k}{\alpha_k-\alpha_j} =$$

$$= \frac{2(1-|\alpha_k|^2)}{\alpha_k^2\sqrt{1-2x\alpha_k+\alpha_k^2}} \prod_{\substack{j=1\\ j\neq k}}^{n-1} \frac{1-\overline{\alpha_j}\alpha_k}{\alpha_k-\alpha_j}.$$

Finally,

$$L_n(x) = \frac{(-1)^{n-1}}{\prod_{k=1}^{n-1}\alpha_k}\left(x + \sum_{j=1}^{n-1} \frac{1-|\alpha_j|^2}{\alpha_j} \right) + \sum_{k=1}^{n-1} \frac{1-|\alpha_k|^2}{\alpha_k^2\sqrt{1-2x\alpha_k+\alpha_k^2}} \prod_{\substack{j=1\\ j\neq k}}^{n-1} \frac{1-\overline{\alpha_j}\alpha_k}{\alpha_k-\alpha_j}. \quad (9)$$

Now we consider the following sequence of parameters. Let $\beta_1, \beta_2, ..., \beta_n$ be a sequence of distinct real positive numbers. Then the parameters $\{\alpha_k\}_{k=0}^{2n+1}$ are as follows: $\alpha_0 = 0$, $\alpha_{2k-1} = \beta_k$, $\alpha_{2k} = -\beta_k$, $k = 1,2,...,n$, $\alpha_{2n+1} = 0$. In this case by the formula (9) we get

$$L_{2n+1}(x) = \frac{x}{\prod_{k=1}^{2n}\alpha_k} + \sum_{k=1}^{2n} \frac{1-|\alpha_k|^2}{\alpha_k^2\sqrt{1-2x\alpha_k+\alpha_k^2}} \prod_{\substack{j=1\\ j\neq k}}^{2n} \frac{1-\overline{\alpha_j}\alpha_k}{\alpha_k-\alpha_j}.$$

These functions are odd, $L_{2n+1}(1) = 1$. Besides using numerical experiment we can draw the graph of some of these functions. In the figures 1, 2, 3 we have the graphs of $L_3(x)$, $L_5(x)$, $L_7(x)$ for $\beta_1 = 0.5$, $\beta_2 = 0.6$, $\beta_3 = 0.7$.

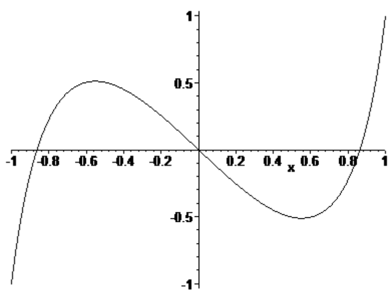

Fig. 1. The graph of $L_3(x)$ for $\beta_1 = 0.5$

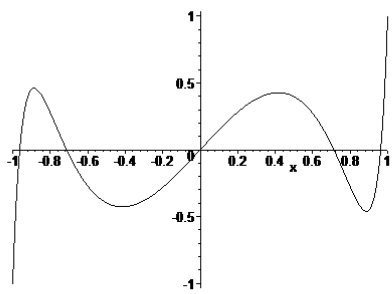

Fig. 2. The graph of $L_5(x)$ for $\beta_2 = 0.6$

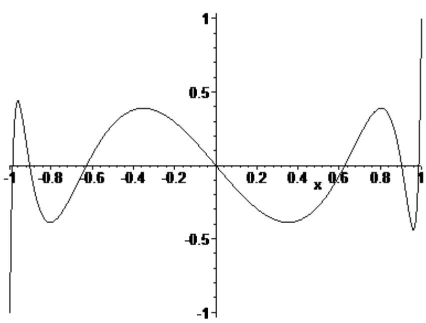

Fig. 3. The graph of $L_7(x)$ for $\beta_3 = 0.7$

Concerning these results some questions can be raised:

(a) Does function $L_{2n+1}(x)$ has $2n+1$ distinct real zeroes in the interval $(-1, 1)$?

(b) Numerical results show that the system $\{L_n(x)\}$ is not orthogonal. Is there a sequence $\{\alpha_n\}$ such that corresponding system $\{L_n(x)\}$ is orthogonal?

Note, that if the answer to the first question is positive than we can use these zeroes as nodes for construction of interpolation polynomial.

References

1. *Suetin P. K.* Classical orthogonal polynomials. Moskow, 1978. (in Russian)

2. *Dzhrbashyan M. M, Kitbalyan A. A.* On one generalization of Chebyshev polynomials // Reports of Armenian SSR Academy of science. 1964. Vol. 38, N 5. P. 263—270. (in Russian)

3. *Rovba E. A.* Orthogonal systems of rational functions on the segment and quadratures of Gauss-type // Mathematica Balkanica. 1999. Vol. 13, N 1—2. P. 187—198.

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